Computer-Generated Holography (CGH)

PHYS4412 (Computational Science II)
Department of Physics & Astronomy
Louisiana State University

I.A One-Dimensional Aperture Oriented Parallel to the Image Screen

General Concept

Consider the amplitude (and phase) of light that is incident at a location y_1 on an image screen that is located a distance Z from a slit of width w. First, as illustrated in Figure I.1, consider the contribution due only to two rays of light: one coming from location Y_1 at the top edge of the slit (a distance D_1 from point y_1 on the screen) and another coming from location Y_2 at the bottom edge of the slit (a distance D_2 from the same point on the screen).

Figure I.1

Aperture Parallel to Image Screen
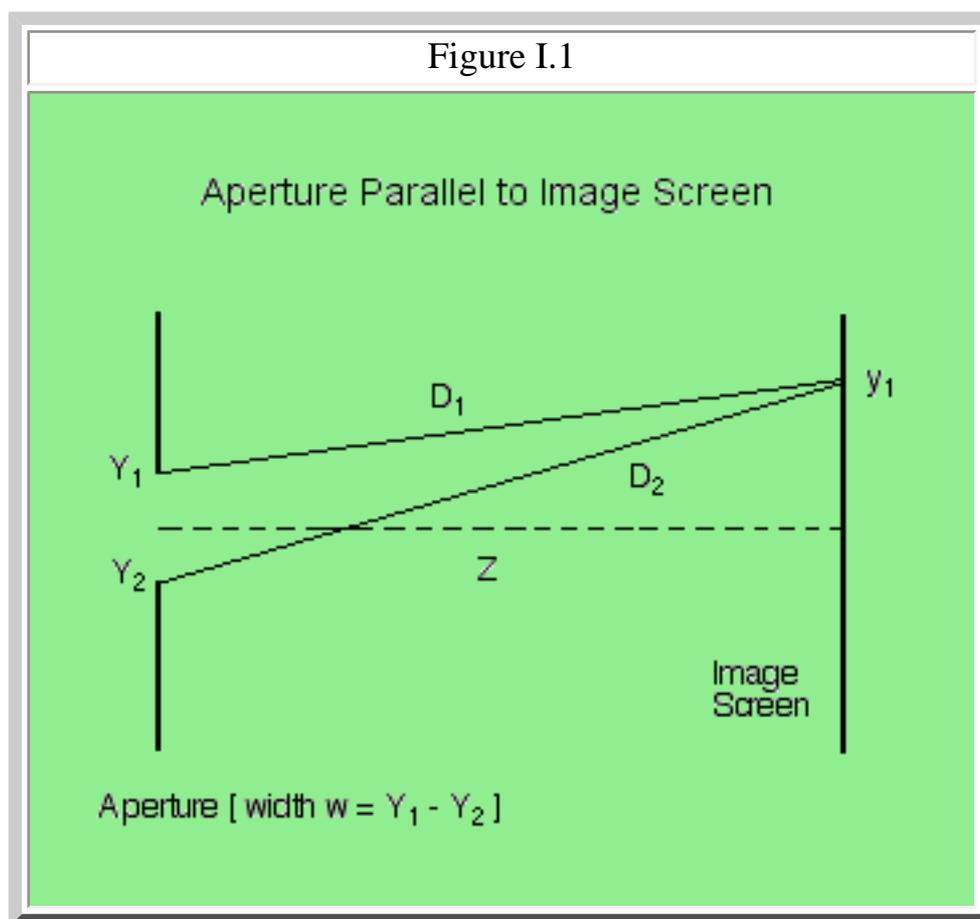
The complex number **A** representing the light amplitude and phase at $y_1$will be,

$$\mathbf{A}\,(y_1) = a(\,Y_1\,)\exp[i(2\pi\,D_1/\,\lambda + \phi_1\,)\,] + a(\,Y_2\,)\exp[i(2\pi\,D_2/\,\lambda + \phi_2\,)\,],$$

[Equation 1]

where λ is the wavelength of the light, $a(\,Y_j\,)$ is the brightness of the light at point Y_j on the aperture, and ϕ_j is the phase of the light as it leaves point Y_j.

Now, if we consider for the moment that at all locations on the aperture Y_j, the light has a brightness $a(\,Y_j\,) = 1.0$ and a phase $\phi_j = 0.0$, then,

$$\mathbf{A}\,(y_1) = A_0\,\{\ 1.0 + \exp[i(2\pi\,/\,\lambda\,)\,(\,D_2 - D_1\,)\,]\ \}\,,$$

[Equation 2]

where $A_0 = \exp[i(2\pi\,D_1\,/\,\lambda\,)\,]$. So the question of whether the amplitude at y_1 on the image screen will be bright due to constructive interference or faint as a result of destructive interference comes down to a question of what the phase difference is between the two distances D_1 and D_2 where,

$$D_j \equiv [\,(\,Y_j - y_1\,)^2 + Z^2\,]^{1/2}$$
$$= [\,Z^2 + y_1^2 - 2\,y_1\,Y_j + Y_j^2\,]^{1/2}$$
$$= L\,[\,1 - (\,2\,y_1\,Y_j\,)/L^2 + Y_j^2/L^2\,]^{1/2}\,,$$

[Equation 3]

and,

$$L \equiv [\,Z^2 + y_1^2\,]^{1/2}\,.$$

[Equation 4]

Utility of FFT Techniques

Now, let's rewrite equation (1) to take into account many more than just two points along the aperture. That is,

$$\mathbf{A}\,(y_1) = \Sigma\,a_j\,\exp[\,i\,(\,2\pi\,D_j/\,\lambda + \phi_j\,)\,]$$
$$= \Sigma\,a_j\,[\,\cos(\,2\pi\,D_j/\,\lambda + \phi_j\,) + i\,\sin(\,2\pi\,D_j/\,\lambda + \phi_j\,)\,]\,,$$

[Equation 5]

or, in the more restrictive case when we assume that everywhere along the aperture the phase $\phi_j = 0$,

$$\mathbf{A}\,(y_1) = \Sigma\, a_j \exp[\, i\,(\, 2\pi\, D_j/\, \lambda\,)\,]$$

$$= \Sigma\, a_j\, [\, \cos(\, 2\pi\, D_j/\, \lambda\,) + i\,\sin(\, 2\pi\, D_j/\, \lambda\,)\,]\,,$$

[Equation 6]

where, in both equations, the summation is taken over all "j" elements of light along the aperture, and to simplify typing we have replaced $a(Y_j)$ with a_j. If $|\, Y_j\, /\, L\,| \ll 1$, we can drop the quadratic term in favor of the linear one in expression (3) and deduce that,

$$D_j \approx L\,[\, 1 - 2\, y_1\, Y_j\, /\, L^2\,]^{1/2}$$

$$\approx L\,[\, 1 - y_1\, Y_j\, /\, L^2\,]\,.$$

[Equation 7]

Hence, equation (6) becomes,

$$\mathbf{A}\,(y_1) \approx A_0\, \Sigma\, a_j \exp\{\, -\, i\,[\, 2\pi\, y_1\, Y_j\, /\, (\, \lambda L\,)\,]\,\}$$

$$= A_0\, \Sigma\, a_j\, \{\, \cos[\, 2\pi y_1\, Y_j\, /\, (\, \lambda L\,)\,] - i\,\sin[\, 2\pi\, y_1\, Y_j\, /\, (\, \lambda L\,)\,]\,\}\,,$$

[Equation 8]

where, now, $A_0 = \exp[i(2\pi\, L\, /\, \lambda\,)\,]$. When written in this form, it should immediately be apparent why discrete Fourier transform techniques (specifically FFT techniques) are useful tools for evaluation of the complex amplitude $\mathbf{A}$.

(If it is not immediately apparent to you how expression [8] can be adapted to FFT techniques, click here for an explicit demonstration.)

Analytical Result

If we consider the limit where the aperture shown in Figure I.1 is divided into an infinite number of divisions, then we can convert the summation in equation (6) or (8) into an integral running between the limits, Y_2 and Y_1. Specifically, equation (8) becomes,

$$\mathbf{A}\,(y_1) \approx A_0\, a_0 \int \exp\{\, -\, i\,[\, 2\pi\, y_1\, Y\, /\, (\, \lambda L\,)\,]\,\}\, dY\,,$$

[Equation 9]

where, again, we have returned to the case where the aperture is assumed to be uniformly bright and set $a(Y) = a_0\, dY$. (It should be understood that a_0 is the brightness of the aperture per unit length.) If

we now make the substitution,

$$\theta \equiv (2 \pi y_1 Y) / (\lambda L) ,$$

[Equation 10]

equation (9) takes the form,

$$\mathbf{A} (y_1) \approx A_0 [a_0 w] (2 \beta_1)^{-1} \int \exp\{ - i \theta \} d\theta ,$$

[Equation 11]

where the lower limit of integration is $\theta_2 = 2 \pi y_1 Y_2 / (\lambda L) = \vartheta_1 - \beta_1$, the upper limit of integration is $\theta_1 = 2 \pi y_1 Y_1 / (\lambda L) = \vartheta_1 + \beta_1$, and

$$\beta_1 \equiv \pi y_1 (\lambda L)^{-1} [Y_1 - Y_2] = \pi y_1 w (\lambda L)^{-1} ,$$

$$\vartheta_1 \equiv \pi y_1 (\lambda L)^{-1} [Y_1 + Y_2] .$$

[Equation 12]

The integral in equation (11) can be readily evaluated, giving,

$$\mathbf{A} (y_1) \approx (- i)^{-1} A_0 [a_0 w] (2 \beta_1)^{-1} [\exp(- i \theta_1) - \exp(- i \theta_2)]$$

$$= i A_0 [a_0 w] (2 \beta_1)^{-1} \exp(- i \vartheta_1) [\exp(- i \beta_1) - \exp(+ i \beta_1)]$$

$$= [A_0 a_0 w \exp(- i \vartheta_1)] [(\sin \beta_1) / \beta_1] .$$

$$= A_0 a_0 w \, e^{(- i \vartheta_1)} \operatorname{sinc} (\beta_1) .$$

[Equation 13]

The last two lines of equation (13) have been derived by realizing that, for any angle χ,

$$e^{(- i \chi)} - e^{(+ i \chi)} = - 2 i \sin(\chi) ,$$

[Equation 14]

and the "sinc" function,

$$\operatorname{sinc}(\chi) \equiv \sin(\chi) / \chi .$$

[Equation 15]

Notice that if the center of the aperture is used to define the origin of the Y (and y) coordinate axis so that $Y_2 = - Y_1$, as illustrated in Figure I.1, then $\vartheta_1 = 0$ and the amplitude $\mathbf{A}$ has no complex component. If the aperture is not centered as depicted in Figure I.1, then the angle ϑ_1 simply serves to introduce a phase shift in the evaluation of $\mathbf{A}$.

Although this problem was solved for a specific location y_1 on the image screen, we see that the sinc function solution can readily be evaluated for any other location on the screen without having to redo the integral.

Sorry, this chapter has not yet been written.

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See also the [list of references](#) supplied by Mark Lucente in his 1994, MIT [Ph.D. Dissertation](#).

Other interesting and useful links regarding CGH:

- The [Spatial Imaging Group](#) within [MIT's Media Lab](#).

  See especially, [Mark Lucente's](#) work on [electro-holography](#) (Holovideo), and publications resulting from his [Ph.D. Dissertation](#).

- [Research into CGH at Nihon University, Japan](#).
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Preface

The concept of computer-generated-holography (CGH) has been of interest to me for a number of years because my astrophysics research group relies heavily upon [animation sequences of complicated volume-rendered images](#) to interpret the results of our large-scale fluid simulations and I am always looking for more effective ways to display/examine these complex three-dimensional structures. In the early 1990s I had the general impression that researchers knew quite well *how* to create holograms digitally from "virtual" computer-generated 3D surfaces but that it had not yet become practical to implement CGH techniques to make, for example, holographic movies because high-resolution digital images would require many many many (!) 2D FFTs. But it also seemed clear to me that with the advent of massively parallel computers (my group was effectively utilizing an 8K-node MasPar at the time), particularly ones that might utilize digital-signal-processing (DSP) hardware, the community might just be crossing a threshold that would make CGH techniques practical.

In 1994, I mentioned to W.O. Hamilton ([LSU Department of Physics & Astronomy](#)) my interest in CGH and he pointed me to the [American Journal of Physics](#) article by [A.E. Macgregor's (1992)](#) --- an article that I now consider to be a superb (perhaps the best!) introduction to the subject of CGH, at least from the perspective of a computational scientist who has relatively little background in Fourier optics. However, my commitment of time to other projects and duties prevented me from looking seriously into CGH techniques and their accompanying computational challenges until this year (1999). And I am quite sure that I would not have found the time even this year to study CGH techniques except for the fact that several students from my [PHYS 2411](#) class requested that I teach a second semester of Computational Science. Because a number of students in that class had impressed me with their native computational skills, I agreed to add PHYS 4412 (Computational Science II) on top of my normal instructional load if students interested in such a class were willing to work with me in a fairly unstructured environment toward a goal of learning *together* how to do CGH.

Well...they agreed, and here we are! Two-thirds of the way through the semester and I think we've figured out how to do it! With this experience behind us, I am convinced more than ever that this field is just about to take off. Virtual reality, here we come!

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April 10, 1999

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I.B Two-Dimensional, Rectangular Aperture
Oriented Parallel to the Image Screen

This discussion parallels the somewhat more detailed one presented in Chapter I.A on the one-dimensional aperture oriented parallel to the image screen.

Utility of FFT Techniques

Consider the amplitude (and phase) of light that is incident at a location (x_1, y_1) on an image screen that is located a distance Z from a rectangular aperture of width w and height h. By analogy with equation (I.A-5), the complex number **A** representing the light amplitude and phase at (x_1, y_1) will be,

$$\mathbf{A}\,(x_1, y_1) = \sum_j \sum_k a_{j,k} \exp[\, i\,(\, 2\pi\, D_{j,k} / \lambda + \phi_{j,k}\,)\,]\,,$$

[Equation 1]

where, here, the summations are taken over all "j,k" elements of light across the entire 2-D aperture, and now the distance $D_{j,k}$ is given by the expression,

$$D_{j,k} \equiv [\,(\,X_j - x_1\,)^2 + (\,Y_k - y_1\,)^2 + Z^2\,]^{1/2}$$
$$= [\,Z^2 + x_1^2 + y_1^2 - 2\,(\,x_1 X_j + y_1 Y_k\,) + X_j^2 + Y_k^2\,]^{1/2}$$
$$= L\,[\,1 - 2\,(\,x_1 X_j + y_1 Y_k\,)/L^2 + (\,X_j^2 + Y_k^2)/L^2\,]^{1/2}\,,$$

[Equation 2]

where,

$$L \equiv [\,Z^2 + x_1^2 + y_1^2\,]^{1/2}\,.$$

[Equation 3]

If $|\,X_j / L\,| \ll 1$ and $|\,Y_k / L\,| \ll 1$, we can drop the quadratic terms in favor of the linear ones in expression (2) and deduce that,

$$D_{j,k} \approx L \, [\, 1 - 2 \, (\, x_1 X_j + y_1 Y_k \,) \, / \, L^2 \,]^{1/2}$$

$$\approx L \, [\, 1 - (\, x_1 X_j + y_1 Y_k \,) \, / \, L^2 \,] \, .$$

[Equation 4]

Hence, equation (1) becomes,

$$\mathbf{A}\,(x_1,y_1) \approx A_0 \sum_j \sum_k a_{j,k} \, \exp\{ i \, \phi_{j,k} \} \, \exp\{ - i \, [\, 2 \, \pi \, (\, x_1 X_j + y_1 Y_k \,) \, / \, (\, \lambda L \,) \,] \, \} \, ,$$

[Equation 5]

where $A_0 = \exp[i(2\pi L / \lambda)]$. When written in this form, it should be apparent why discrete Fourier transform techniques (specifically 2D-FFT techniques) are useful tools for evaluation of the complex amplitude **A**.

Analytical Result

Now, as in [Chapter I.A](), we would like to consider the case where the rectangular aperture is divided into an infinite number of divisions in both the X and Y dimensions and convert the summations in equation (5) into integrals whose limits in both directions are given by the edges of the aperture. If we specifically consider the case where the aperture is assumed to be uniformly bright (*i.e.*, $a_{j,k} = a_0 \, dX \, dY$, and a_0 is the brightness per unit area), and the phase $\phi = 0$ at all location on the aperture, equation (5) becomes,

$$\mathbf{A}\,(x_1,y_1) \approx A_0 \, a_0 \int \int \exp\{ - i \, [\, 2\pi \, (\, x_1 X + y_1 Y \,) \, / \, (\, \lambda L \,) \,] \, \} \, dX \, dY \, ,$$

$$= A_0 \, a_0 \int \exp\{ - i \, [2\pi \, x_1 X \, / \, (\lambda L)] \} \, dX \int \exp\{ - i \, [2\pi \, y_1 Y \, / \, (\lambda L)] \} \, dY \, .$$

[Equation 6]

Both of these integrals can be completed in the same fashion as described in [Chapter I.A]() for the 1D slit, giving,

$$\mathbf{A}\,(x_1,y_1) \approx A_0 \, a_0 \, (w \, h) \, e^{(- i \, \Theta_1)} \, e^{(- i \, \vartheta_1)} \, \text{sinc}\,(\, \alpha_1 \,) \, \text{sinc}\,(\, \beta_1 \,) \, .$$

[Equation 7]

where,

$$\alpha_1 \equiv \pi \, x_1 \, (\, \lambda L \,)^{-1} \, [\, X_1 - X_2 \,] = \pi \, x_1 \, h \, (\, \lambda L \,)^{-1} \, ,$$

$$\Theta_1 \equiv \pi \, x_1 \, (\lambda L)^{-1} \, [\, X_1 + X_2 \,] \, ,$$

$$\beta_1 \equiv \pi \, y_1 \, (\lambda L)^{-1} \, [\, Y_1 - Y_2 \,] = \pi \, y_1 \, w \, (\lambda L)^{-1} \, ,$$

$$\vartheta_1 \equiv \pi \, y_1 \, (\lambda L)^{-1} \, [\, Y_1 + Y_2 \,] \, .$$

<div align="center">[Equation 8]</div>

It is worth noting that this derivation closely parallels the one presented in § 8.5.1 (p. 393) of Born & Wolf. Specifically, our equation (6) is identical to the "Fraunhofer diffraction integral" written down by Born & Wolf at the beginning of their § 8.5.1; and if we follow Born & Wolf's lead and position the origin of our coordinate system at the center of the rectangle, then $\Theta_1 = \vartheta_1 = 0$, and the intensity I(P) at point P given by their eq. (1) precisely matches the expression for $\mathbf{A}^2$ that is obtained using our equation (7).


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I.C Relevance to Holograms

Okay, so what do Chapters [I.A](#) and [I.B](#) have to do with the numerical construction of a hologram? Well, this is best explained by borrowing an illustration from the [Nishikawa *et al.* (1997)](#) paper (redrawn here as Figure I.2), and comparing it to [Figure I.1](#) from Chapter I.A.

Figure I.2
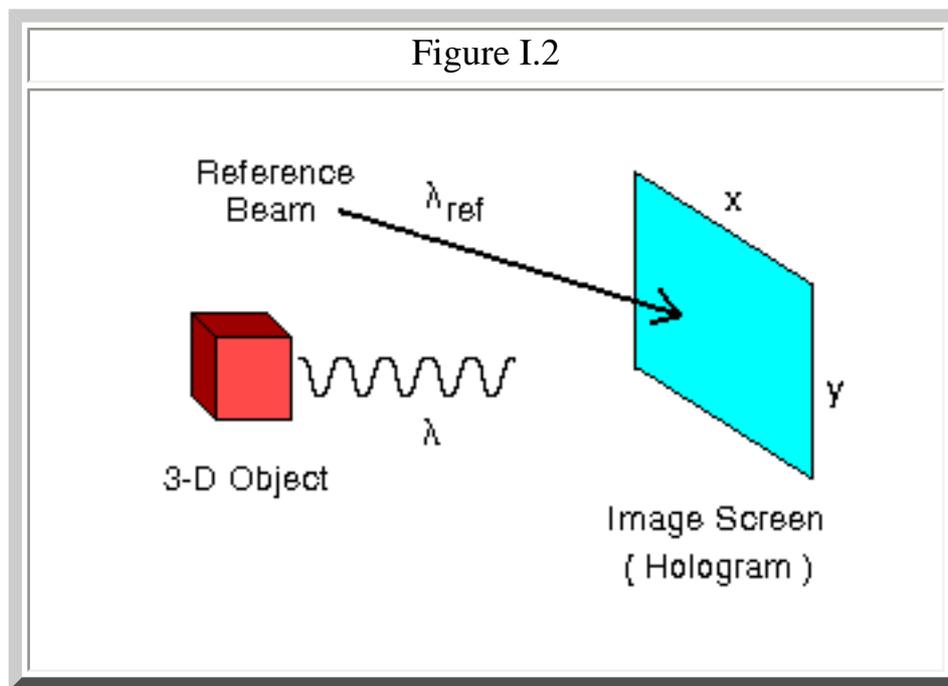
In general, a hologram is made by combining light of wavelength λ that is coming from the surface of a 3-dimensional object with coherent light from a reference beam of wavelength λ_{ref}. The interference pattern that is created on the "image screen" via the combination of these two separate sources of light is recorded (usually in the form of an exposed piece of photographic film) as the hologram. Let's consider the behavior of these two sources of light, separately.

Light from the Object

The discussions in Chapters [I.A](#) and [I.B](#) illustrate how light that originates from an extended object

(aperture) illuminates various points along the image screen. For example, from the perspective of the image screen, the front vertical edge of the red cube shown above in Figure I.2 looks like a vertical, one-dimensional "slit" aperture of width "w", as illustrated in <u>Figure I.1</u>. Hence, the complex amplitude of light that is striking various points "y" along any vertical column of the image screen (hologram) *due solely to the light coming from this edge of the cube* is given by expression <u>(I.A-13)</u>, that is,

$$\mathbf{A} (y) \approx A_0 \, a_0 \, w \, \text{sinc} (\beta) ,$$

[Equation 1]

where,

$$\beta = \pi \, y \, w \, (\lambda L)^{-1} ,$$
$$L = [Z^2 + y^2]^{1/2} ,$$

[Equation 2]

Z is the shortest distance measured between the front vertical edge of the cube and the selected column of the image screen, and for purposes of illustration, we have set to zero the offset phase angle ϑ_1 that appears in <u>equation (I.A-13)</u>.

However, as depicted in Figure I.2, the image screen is not being illuminated by just one vertical edge of the cube but, instead, by one entire two-dimensional face of the cube (the side of the cube that is facing to the right and is therefore not visible in the figure). From the perspective of the image screen (hologram), the surface of the cube looks like a two-dimensional aperture of width "w" and height "w." Therefore, the more complete expression for the complex amplitude of light striking the image screen is,

$$\mathbf{A} (x, y) \approx A_0 \, a_0 \, w^2 \, \text{sinc} (\alpha) \, \text{sinc} (\beta) ,$$

[Equation 3]

where,

$$\alpha = \pi \, x \, w \, (\lambda L)^{-1} ,$$
$$\beta = \pi \, y \, w \, (\lambda L)^{-1} ,$$
$$L = [Z^2 + x^2 + y^2]^{1/2} ,$$

[Equation 4]

as given by <u>equations (I.B-7) and (I.B-8)</u>. (Note that, again for the sake of illustration, we have set the two offset phase angles $\Theta_1 = \vartheta_1 = 0$.)

Light from the Reference Beam

The complex number **R** representing the light amplitude and phase of the reference beam at any

position (x, y) on the image screen is, simply,

$$\mathbf{R}(x, y) = a_{ref} \exp\{ -i\, \phi_{ref} \}\,.$$

[Equation 5]

It is customary to consider the reference beam to be a laser that is capable of producing a coherent plane wave which is broad enough in extent to easily illuminate the entire image screen. Then, if the propagation vector $\mathbf{k}_{ref}$ of the reference plane wave is tilted about the y-axis of the image screen at an angle ζ_{ref} with respect to the unit vector $\mathbf{n}_{screen}$ that is normal to the image screen,

$$\phi_{ref} = 2\pi\, x \sin(\zeta_{ref}) / \lambda_{ref}\,.$$

[Equation 6]

Combined Light Sources

Combining these two sources of light, the complex amplitude of light hitting the image screen (hologram) will be,

$$\mathbf{W}(x, y) = \mathbf{A}(x, y) + \mathbf{R}(x, y)\,,$$

[Equation 7]

or, since the intensity of the light $\mathbf{I}(x,y)$ that gets recorded on the film (hologram) is actually the magnitude of this complex amplitude, we have,

$$\mathbf{I}(x, y) = \mathbf{W}\,\mathbf{W}^{*}$$
$$= \mathbf{A}\,\mathbf{A}^{*} + \mathbf{R}\,\mathbf{R}^{*} + 2\,(\,\mathbf{A}_{Re}\,\mathbf{R}_{Re} + \mathbf{A}_{Im}\,\mathbf{R}_{Im}\,)\,.$$

[Equation 8]

As Trestor (1996) and Nishikawa *et al.* (1997) both clearly explain, it is the last terms on the right-hand-side of equation (8) involving cross-products of $\mathbf{A}$ and $\mathbf{R}$ that play the most important role in holography. When the intensity of light produced by these terms is properly recorded on a piece of transparent film, a virtual image of the original object can be retrieved by shining the original reference beam back through the film at the proper angle ζ_{ref} relative to the surface of the film. We shall say more about this important issue in Chapter I.D.

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I.D Caution and Words of Wisdom

As Nishikawa *et al.* (1997) have indicated [see their equation (5)], the most general expression for the complex amplitude of light that arrives at a given (x,y) position on the image screen from a three-dimensional object whose surface (aperture) is defined by "n" separate, infinitesimal points of light, is,

$$\mathbf{A}(x, y) = \sum_n a_n \exp[\, i\, (\, 2\pi\, D_n/\lambda + \phi_n\,)\,] \, ,$$

[Equation 1]

where,

$$D_n \equiv [\,(\, X_n - x\,)^2 + (\, Y_n - y\,)^2 + Z_n^2\,]^{1/2} \, .$$

[Equation 2]

When the aperture (object surface) is a rectangle that is parallel to the image screen, it is wisest to place the "n" points across the surface in such a way that they form an ordered, 2D rectilinear array having "j" columns and "k" (= n / j) rows, in which case these two equations immediately take the form of equations (I.B-1) and (I.B-2), respectively.

Potentially Dangerous Simplifications

In deriving the results reported in Chapters I.A, I.B, and I.C, we generally have employed the following three broad, simplifying assumptions.

1. The linear dimensions of the aperture (object) are very small compared to the distance between the aperture (object) and the image screen;

2. The aperture (object surface) is uniformly bright; and

3. At all points across the aperture (object surface), the phase of the light is zero as it leaves the aperture (object surface).

By adopting the first of these assumptions, we have been able to drop the nonlinear terms in the definition of D_n and then rewrite equation (1) (or its equivalent) in a form that can be solved using standard 1D- or 2D-FFT techniques. This makes the determination of **A** both straightforward and relatively fast. But it must be understood that a result obtained in this manner is only an approximate one because the nonlinear terms in D_n have been ignored.

When all three of these assumptions are adopted, we have been able to rewrite equation (1) in a form that can be integrated analytically. This, of course, can speed up the determination of **A** tremendously, but again one has to appreciate that the obtained result is only an approximation. Throughout the next few chapters we will continue to utilize these three simplifying assumptions in an effort to demonstrate how the amplitude **A**(x,y) can be straightforwardly calculated even when a flat, rectangular aperture (*e.g.,* the surface of a cube) is tilted with respect to the image screen and/or when the aperture (surface of an object) has a triangular shape. These two concepts are critical ones as we develop a technique that will permit the rapid numerical construction of holographic images for arbitrarily shaped and arbitrarily positioned, three-dimensional objects. In later chapters we shall return to a discussion of these enumerated simplifying assumptions in order to ascertain what degree of accuracy and what object features must be sacrificed in order to achieve palatable computational speeds.

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**II.A** One-Dimensional Aperture, Tilted with Respect to the Image Screen

In Chapter I, we considered only apertures that were oriented parallel to the image screen. This is appropriate for determining the holographic image of an object in only very limited situations; for example, it applies to the red cube shown in in [Figure I.2](#), but only because the cube is oriented such that one of its faces is parallel to the image screen. For practically any other orientation, three separate faces of the cube will be visible from every point on the image screen and, although they each can be described as a flat, square aperture, each of the apertures will be inclined to the screen. Hence, we must learn how to evaluate the amplitude and phase of light which hits the image screen from an aperture that is inclined to the screen.

General Concept

Consider the complex amplitude of light $A(y_1)$ that is incident at a location y_1 along an image screen whose center is located a distance Z from a slit of width w, but that is tilted at an angle ψ with respect to the aperture, as illustrated in Figure II.2.

Figure II.2

Aperture tilted at angle ψ to Image Screen

Aperture [width $w = Y_1 - Y_2$]
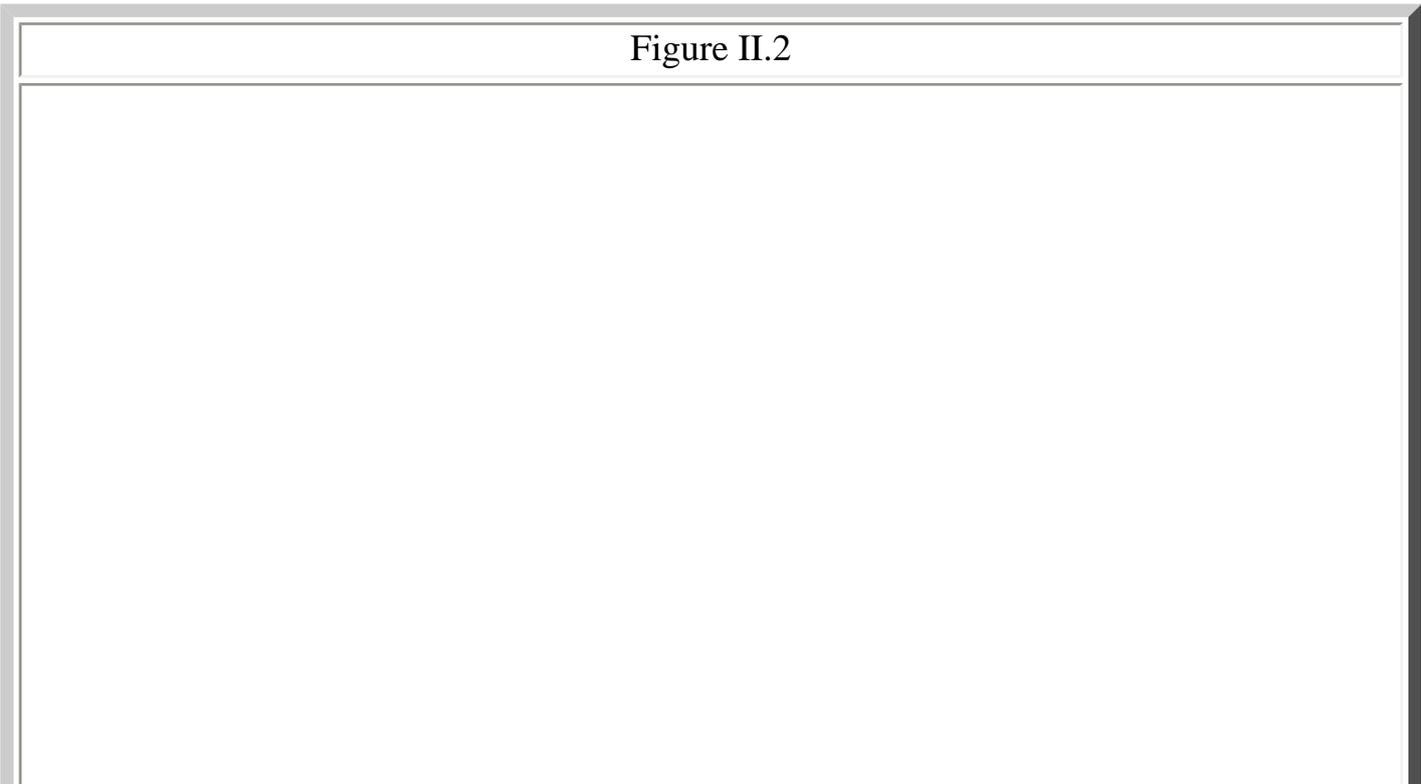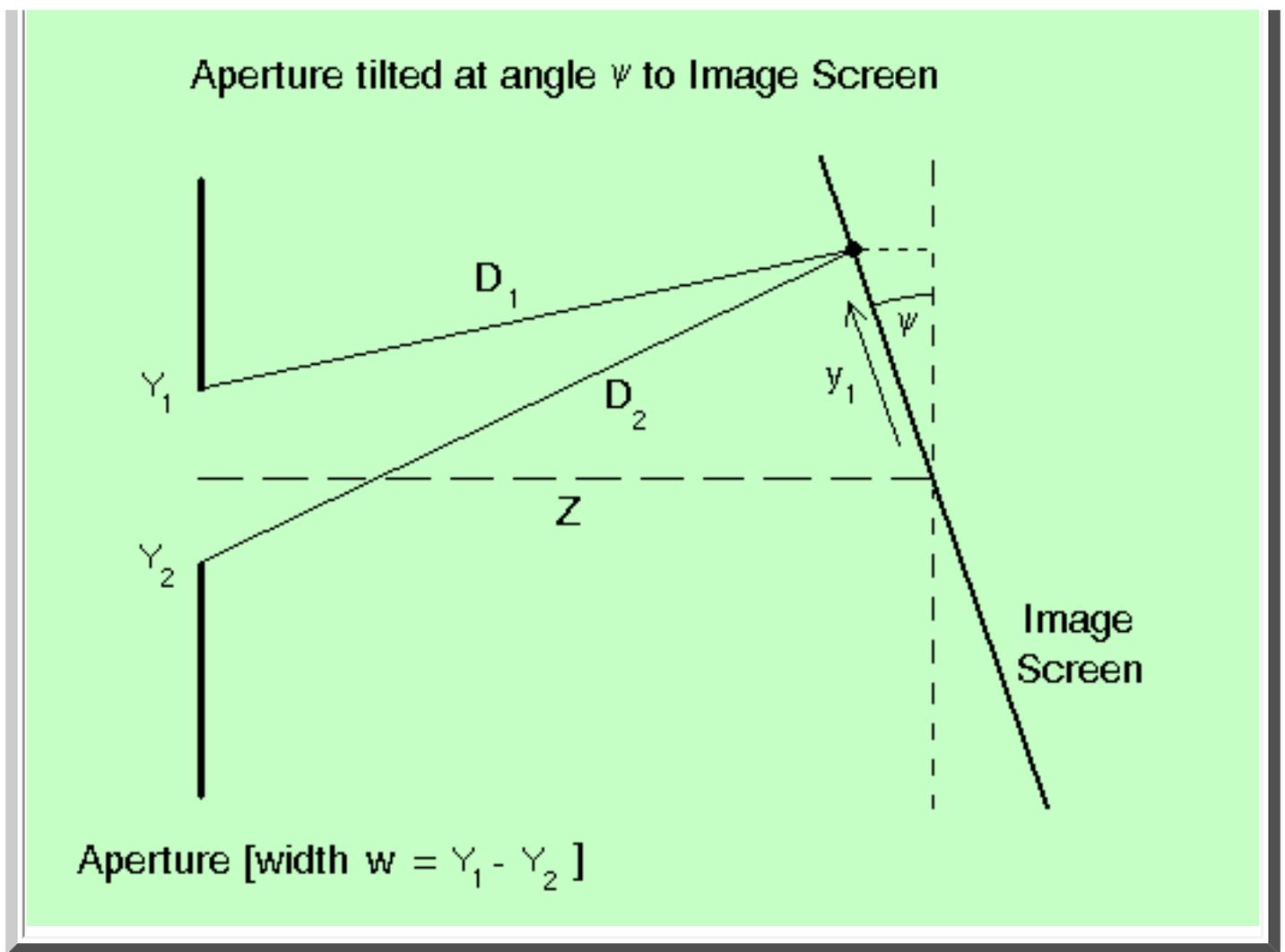
[It might seem more natural to continue to draw the image screen vertically and then to tilt the aperture with respect to the screen's coordinate system, as has been done by Leseberg & Frère (1988) and Patorski (1983). But as we shall demonstrate, it actually proves to be much easier to understand and to generalize our derived results if they are cast first in terms of a coordinate system that is tied in a natural way to the aperture, as shown here.]

If we again consider the case where the phase $\phi = 0$ at all locations along the slit, the expression for $A(y_1)$ is given by equation (6) from Chapter I.A , namely,

$$\mathbf{A}(y_1) = \sum a_j \exp[\, i\, (\, 2\pi\, D_j / \lambda\,)\,]$$

[Equation 1]

where, as before, the summation is taken over all "j" elements of light along the aperture, but now,

$$D_j = [\, (\, Y_j - y_1 \cos\psi\,)^2 + (\, Z - y_1 \sin\psi\,)^2\,]^{1/2}$$

$$= L\, [\, 1 - (\, 2\, y_1 \cos\psi\, Y_j\,)/L^2 + Y_j^2/L^2\,]^{1/2}$$

[Equation 2]

and,

$$L \equiv [\,(\,Z - y_1 \sin\psi\,)^2 + (\,y_1 \cos\psi\,)^2\,]^{1/2}$$
$$= [\,Z^2 + y_1^2 - 2\,Z\,y_1 \sin\psi\,]^{1/2}\,.$$

[Equation 3]

If $|\,Y_j\,/\,L\,| \ll 1$, we can drop the quadratic term in favor of the linear one in expression (2) and deduce that,

$$D_j \approx L\,[\,1 - (\,y_1 \cos\psi\,Y_j\,)\,/\,L^2\,]\,.$$

[Equation 4]

Hence, equation (1) becomes,

$$\mathbf{A}\,(y_1) \approx A_0 \sum a_j \exp\{\,-\,i\,[\,2\pi\,(\,y_1 \cos\psi\,Y_j\,)\,/\,(\,\lambda L\,)\,]\,\}\,,$$

[Equation 5]

where, as before, $A_0 = \exp[\,i(2\pi\,L\,/\,\lambda\,)\,]$.

Analytical Result

If, as in Chapter I.A, we consider the limit where the aperture shown in Figure 2 is divided into an infinite number of divisions, then we can convert the summation in equation (5) into an integral running between the limits, Y_2 and Y_1. Specifically, equation (5) becomes,

$$\mathbf{A}\,(y_1) \approx A_0\,a_0 \int \exp\{\,-\,i\,[\,2\pi\,(\,y_1 \cos\psi\,Y\,)\,/\,(\,\lambda L\,)\,]\,\}\,dY\,.$$

[Equation 6]

Following the same sequence of steps as in Chapter I.A, this integral can be completed analytically to give,

$$\mathbf{A}\,(y_1) \approx A_0\,a_0\,w\,e^{(\,-\,i\,\vartheta_1\,)}\,\mathrm{sinc}(\,\beta_1\,)\,,$$

[Equation 7]

where, here,

$$\beta_1 \equiv \pi\,y_1 \cos\psi\,(\,\lambda\,L\,)^{-1}\,[\,Y_1 - Y_2\,] = \pi\,y_1 \cos\psi\,w\,(\,\lambda\,L\,)^{-1}\,,$$

$$\vartheta_1 \equiv \pi \, y_1 \cos\psi \, (\lambda L)^{-1} [Y_1 + Y_2] .$$

[Equation 8]

Hence, the *form* of the solution for **A** is identical to its counterpart in the case when the slit is oriented parallel to the image screen, but the actual result is different because, the angle ψ enters into the calculation of β_1, ϑ_1, and L.

Discussion

Before we head into a discussion of tilted, two-dimensional apertures, it is worth noting that the result presented here in the form of equations (7) and (8) could have been derived by inspection from equations (I.A-13) and (I.A-12), respectively, had we been clever enough to appreciate what was actually being derived. Although we have presented the derivations in this chapter as well as in Chapter I.A in the context of wanting to know the complex amplitude of light that falls across an entire (geometrically flat) image screen, in reality what we have done, first, is derive the complex amplitude of light that has collected *at one particular point* in space and then realized that the analytical result also works for arbitrary locations along the (flat) screen.

If, for the moment, we focus back on the single point, we see that in both chapters the quantity "L" that enters into the calculation of β_1 and ϑ_1 is simply the distance measured from the origin of the aperture to the point in question which, in terms of the Cartesian coordinate system (x_a, y_a, z_a) that is affixed to the aperture is,

$$L = [z_a^2 + y_a^2]^{1/2}.$$

[Equation 9]

Furthermore, in both cases, the angles β_1 and ϑ_1 are obtained by multiplying a particular wave number (respectively, k_β and, k_ϑ) by the point's "vertical" location y_a, as measured in the aperture coordinate system. That is to say,

$$\beta_1 = k_\beta \, y_a,$$
$$\vartheta_1 = k_\vartheta \, y_a,$$

[Equation 10]

where,

$$k_\beta \equiv \pi \, (\lambda L)^{-1} [Y_1 - Y_2] ,$$
$$k_\vartheta \equiv \pi \, (\lambda L)^{-1} [Y_1 + Y_2] .$$

[Equation 11]

So, the results of Chapter I.A can be obtained from equations (7), and (9)-(11), above, by realizing that when the image screen is aligned parallel to the aperture (as shown in [Figure I.1](#)), $y_a = y_1$ and $z_a = Z$. And the results of this chapter can be obtained from the same set of equations by realizing that when the image screen is tilted with respect to the aperture (as shown in [Figure II.2](#)), $y_a = y_1\cos\psi$ and $z_a = Z - y_1\sin\psi$. With this understanding, it should now be possible to immediately calculate the complex amplitude **A** at any group of points in space, relative to the aperture -- even along an arbitrarily curved image screen.

Computer-Generated Holography (CGH)

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II.B Two-Dimensional, Rectangular Aperture Tilted (about the x-axis)
at an angle ψ relative to the Image Screen

This discussion combines details from Chapter I.B in which we discussed a two-dimensional aperture oriented parallel to the image screen, and Chapter II.A in which a one-dimensional aperture was tilted with respect to the image screen.

General Concept

Consider the amplitude (and phase) of light that is incident at a location (x_1,y_1) on an image screen whose center is located a distance Z from a rectangular aperture of width w and height h but which is tilted about its x-axis by an angle ψ relative to the image screen. By analogy with equations (I.B-1) and (II.A-2), the complex number **A** representing the light amplitude and phase at (x_1,y_1) will be,

$$\mathbf{A}(x_1,y_1) = \sum_j \sum_k a_{j,k} \exp[\, i\, (\, 2\pi\, D_{j,k}/\, \lambda + \phi_{j,k}\,)\,]\, ,$$

[Equation 1]

where, now, the distance $D_{j,k}$ is given by the expression,

$$D_{j,k} \equiv [\, (\, X_j - x_1\,)^2 + (\, Y_k - y_1\cos\psi\,)^2 + (\, Z - y_1\sin\psi\,)^2\,]^{1/2}$$

$$= \{\, [\, (Z - y_1\sin\psi)^2 + x_1{}^2 + (y_1\cos\psi)^2\,] - 2\, (\, x_1 X_j + y_1\cos\psi\, Y_k\,) + X_j{}^2 + Y_k{}^2\, \}^{1/2}$$

$$= L\, [\, 1 - 2\, (\, x_1 X_j + y_1\cos\psi\, Y_k\,)\, /\, L^2 + (\, X_j{}^2 + Y_k{}^2)\, /\, L^2\,]^{1/2}\, ,$$

[Equation 2]

and,

$$L \;\equiv\; \{\, (\, Z - y_1\sin\psi\,)^2 + x_1{}^2 + (\, y_1\cos\psi\,)^2\, \}^{1/2}$$

$$= [\, Z^2 + x_1{}^2 + y_1{}^2 - 2\, Z\, y_1\sin\psi\,]^{1/2}\, .$$

[Equation 3]

Comparison with Leseberg & Frère

Here we depart somewhat from the sequence of steps that have been taken in earlier chapters in order to compare our result for the tilted 2-D, rectangular aperture with the published result of Leseberg & Frère (1988). Considering the situation when $|X_j / L| << 1$ and $|Y_k / L| << 1$, we will use the binomial expansion to approximate the square root in equation (2), but will not yet drop the quadratic terms in favor of the linear ones. That is, we will for the moment employ the approximation,

$$D_{j,k} \approx L \left[1 - (x_1 X_j + y_1 \cos\psi\, Y_k) / L^2 + (1/2)(X_j^2 + Y_k^2) / L^2 \right].$$

[Equation 4]

With this approximation, equation (1) becomes,

$$\mathbf{A}(x_1, y_1) \approx A_0 \sum_j \sum_k a_{j,k} \exp\{ i\, \phi_{j,k} \} \exp\{ i\, [\pi/(\lambda L)] [(X_j^2 + Y_k^2) - 2(x_1 X_j + y_1 \cos\psi Y_k)] \},$$

[Equation 5]

where $A_0 = \exp[i(2\pi L / \lambda)]$. Now, as in Chapter I.B, we would like to consider the case where the rectangular aperture is divided into an infinite number of divisions in both the X and Y dimensions and convert the summations in equation (5) into integrals whose limits in both directions are given by the edges of the aperture. Without assuming that the aperture is uniformly bright [*i.e.*, setting $a_{j,k} = a_0(X,Y)\, dX\, dY$], and by not yet prescribing the phase ϕ across the aperture, equation (5) becomes,

$$\mathbf{A}(x_1, y_1) \approx A_0 \int \int a_0(X,Y) \exp\{ i\, \phi(X,Y) \} \exp\{ i\, [\pi/(\lambda L)] [(X^2 + Y^2)$$
$$- 2(x_1 X + y_1 Y \cos\psi)] \}\, dX\, dY.$$

[Equation 6]

In order to see the correspondence between this expression and the Leseberg & Frère (1988) discussion, it is important to realize that Leseberg & Frère used a slightly different coordinate notation than has been used here in the derivation of equation (6). As illustrated earlier in our Figure 2, we have chosen to measure the separation (Z) between the aperture and the image screen along a coordinate axis that runs through the center of the aperture and lies *perpendicular to the aperture*, whereas Leseberg & Frère have measured this separation (they called it z_0) along a coordinate axis that runs through the center of the aperture but lies *perpendicular to the image screen*. Hence, the following transformations must be used:

Leseberg & Frère		This derivation
x	=	x_1
y	=	$y_1 - Z \sin\psi$
z_0	=	$Z \cos\psi$

It is also necessary to utilize the following notational relationships:

Leseberg & Frère		This derivation
r	=	D
r_0	=	L
u_0	=	a_0
x'	=	X
y'	=	Y

Making these substitutions, equation (6) becomes,

$$\mathbf{A}\,(x,y) \approx A_0 \int \ \int \ u_0(x',y') \exp\{i\,\phi(x',y')\,\} \exp\{\,i\,[\pi/(\lambda r_0)]\,[\,(x'^2 + y'^2)$$
$$- 2(xx' + yy'\cos\psi + z_0 y'\sin\psi\,)\,]\,\}\,dx'\,dy'\,.$$

<div align="center">[Equation 7]</div>

Finally, we note that Leseberg and Frère considered the specific case where "the object is illuminated by an on-axis plane wave,"[1] and in so doing, adopted the following phase function $\phi(x',y')$ for the light leaving the aperture:

$$\phi(x',y') = (\,2\pi\,/\,\lambda\,)\,y'\sin\psi$$
$$= [\,2\pi\,/\,(\,\lambda\,r_0\,)\,]\,r_0\,y'\sin\psi\,.$$

[Equation 8][1]

In this particular case, then, equation (7) becomes,

$$\mathbf{A}\,(x,y) \approx A_0 \int \int u_0(x',y')\, \exp\{\,i\,[2\pi/(\lambda)\,y'\,\sin\psi]\,\}\, \exp\{\,i\,[\pi/(\lambda r_0)]\,[\,(x'^2 + y'^2)\,]\,\}$$

$$\exp\{\,-\,i\,[\,2\pi/(\lambda r_0)]\,[xx' + yy'\cos\psi + z_0\,y'\,\sin\psi\,]\,\}\,dx'\,dy'$$

$$= A_0 \int \int u_0(x',y')\, \exp\{\,i\,[\pi/(\lambda r_0)]\,[\,(x'^2 + y'^2)\,]\,\}$$

$$\exp\{\,-\,i\,[\,2\pi/(\lambda r_0)]\,[xx' + yy'\cos\psi + (\,z_0 - r_0\,)\,y'\,\sin\psi\,]\,\}\,dx'\,dy'\ ,$$

[Equation 9]

which is identical to equation (2) of Leseberg & Frère (1988).

Now the principal point of this derivation has been to demonstrate that the *form* of our integral expression (6) [or of Leseberg & Frère's equivalent expression shown here as equation (9)], is the same as the form of equation (I.B-6) for the untilted rectangular aperture. That is, in both cases, the last exponential term under the integrals can be written as,

$$\exp\{\,-\,i\,[\,2\pi/(\lambda L)]\,[\,\nu\,X + \mu\,Y\,]\,\}\,dX\,dY\ ,$$

[Equation 10]

where ν and μ are coefficients that do not depend on X or Y (*i.e.,* on the intrinsic shape or extent of the aperture). Hence, as Leseberg & Frère put it, "... the Fourier transform and the coordinate transformation can be implemented independent of each other." A similar point has been made by Rabal, Bolognini, & Sicre (1985), although their comments were targeted towards experimental applications. As our discussion at the end of Chapter II.A points out, this idea can even begin to be appreciated in the context of a one-dimensional slit.

Discussion

Coordinate Transformation Independence

The discovery that the Fourier transform and the coordinate transformation can be implemented independent of each other is an extremely important result. It means that, once the holographic pattern produced by a *specific* aperture has been calculated once, you should be able to "fly around that aperture," viewing it from any other direction in space (any other angle ψ) without having to recalculate the holographic pattern.

It does not appear as though Leseberg & Frère actually viewed their result from this perspective, however. They were not looking for ways to speed up the calculation of a "fly-by" sequence, but rather for ways to speed up the calculation of the holographic pattern in the first place! What they discussed, therefore, were ways in which the extended surface of a nonsimple structure might be broken down into a series of (presumably connected) small, rectangular apertures that are each oriented at a different angle to the image screen.

Here we will complete the double integral in equation (6) analytically and offer *that* result as a mechanism for speeding up the calculation of the holographic pattern, then return to the Leseberg & Frère result later as a mechanism to implement rapid "fly-by" capabilities once multiple apertures have been pieced together to construct the surface of a three-dimensional object.

Analytical Prescription

If we drop from equation (6) [or eq. (7) or eq. (9)] the terms that are quadratic in X (= x') and Y (= y') in favor of the terms that are linear in these variables, respectively, and assume that the aperture is uniformly illuminated (*i.e.,* set $u_0 = a_0 \cos\psi$ = constant),[2] then the integrals can be completed analytically. (As Leseberg & Frère [1988] point out, even when the Fourier transform is completed numerically, "it is usual to neglect the additional quadratic phase factor [on the image screen]." If the intensity across the image screen is then turned into a transparent hologram, when light is shown back through the hologram, "the complex conjugate of this phase appears in the reconstructed image." But, as Leseberg & Frère put it, "this is of no concern for an intensity recording as performed by most detecting devices.")

Comparison with Patorski (1983)

Employing these approximations in equation (9), for example, the expression to be integrated is,

$$\mathbf{A}(x,y) \approx A_0\, a_0\, \cos\psi \int \int \exp\{ i\, [2\pi/(\lambda)\, y'\, \sin\psi] \}$$
$$\exp\{ -i\, [\, 2\pi/(\lambda r_0)]\, [xx' + yy'\, \cos\psi + z_0\, y'\, \sin\psi\,] \}\, dx'\, dy' .$$

[Equation 11]

Using the shorthand notation, $k \equiv 2\pi/\lambda$, $p \equiv x/r_0$, and $q \equiv y/r_0$, equation (11) may also be written in the form,

$$\mathbf{A}(x,y) \approx A_0\, a_0\, \cos\psi \int \int \exp\{ i\, [k\, y'\, \sin\psi] \}$$
$$\exp\{ -i\, k\, [px' + qy'\, \cos\psi + (z_0/r_0)\, y'\, \sin\psi\,] \}\, dx'\, dy' ,$$

[Equation 12]

which is precisely the same as Patorski's (1983) equation (2).[3] Finally, writing this equation in the form of expression (10), we obtain,

$$\mathbf{A}\,(x,y) \approx A_0\, a_0\, \cos\psi \int \int \exp\{ - i\, k\, [px' + y'(q\cos\psi + (z_0/r_0)\sin\psi - \sin\psi\,)\,]\,\} \,dx'\,dy'$$

$$= A_0\, a_0\, \cos\psi \int \exp\{ - i\, \nu_P\, x'\,\} \,dx' \int \exp\{ - i\, \mu_P\, y'\,\} \,dy'\,.$$

[Equation 13]

where,

$$\nu_P \equiv kp\,,$$

$$\mu_P \equiv k\,[q\cos\psi + (z_0/r_0)\sin\psi - \sin\psi\,]\,,$$

[Equation 14]

and the subscript "P" is being used to indicate that these are the wave numbers derived by Patorski (1983). Now, as in Chapter I.B, it seems clear that the limits of integration should be from X_2 to X_1 in the x' (= X) coordinate, and in the y' (= Y) coordinate the limits should be from Y_2 to Y_1. Employing these limits, then, both of these integrals can be completed in the same fashion as described in Chapter I.A for the 1D slit, giving,

$$\mathbf{A}\,(x,y) \approx A_0\, a_0\, (w\,h)\, \cos\psi\, e^{(\,-\,i\,\Theta_1\,)}\, e^{(\,-\,i\,\vartheta_1\,)}\, \mathrm{sinc}(\,\alpha_1\,)\,\mathrm{sinc}(\,\beta_1\,)\,,$$

[Equation 15]

where,

$$\alpha_1 \equiv \nu_P\,[\,X_1 - X_2\,]\,/\,2 = \nu_P\, h\,/\,2\,,$$

$$\Theta_1 \equiv \nu_P\,[\,X_1 + X_2\,]\,/\,2\,,$$

$$\beta_1 \equiv \mu_P\,[\,Y_1 - Y_2\,]\,/\,2 = \mu_P\, w\,/\,2\,,$$

$$\vartheta_1 \equiv \mu_P\,[\,Y_1 + Y_2\,]\,/\,2\,.$$

[Equation 16]

Assuming the aperture is centered on the image screen -- *i.e.,* $Y_2 = -\,Y_1$ and $X_2 = -\,X_1$ -- the two phase angles $\Theta_1 = \vartheta_1 = 0$, and equation (15) becomes,

$$\mathbf{A}\,(x,y) \approx A_0\, a_0\, (w\,h)\, \cos\psi\, \mathrm{sinc}(\,\alpha_1\,)\,\mathrm{sinc}(\,\beta_1\,)\,,$$

[Equation 17]

which is identical to Patorski's (1983) equation (6) in all but one respect. In the y' (= Y) coordinate

direction, Patorski set his integration limit to $Y_1 = (w/2)\cos\psi$ whereas we have set ours to $Y_1 = (w/2)$. Hence the argument of the second sinc function in Patorski's expression differs from ours by a factor of $\cos\psi$. We believe that our limits are the correct ones because, independent of the direction from which you look, the integration is done in the coordinate system of the aperture and therefore should always have the same limits. This aspect of the problem is, in part, what permits us to implement the Fourier transform and the coordinate transformation independent of one another, as discussed above. As explained in [footnote 2](), below, we agree with Patorski that a factor of $\cos\psi$ should enter the calculation when considering the brightness per unit area of the light that is incident on the aperture, but this factor should not be introduced into the limits of integration because tilting the aperture does not alter the aperture's inherent size or shape.

A Formulation More Appropriate for CGH

Having shown the correspondence between our derivation and similar ones presented previously by [Leseberg & Frère (1988)]() and [Patorski (1983)](), we now return to our original notation in [equation (6)](), and apply the same conditions regarding the property of light leaving the surface of the aperture as we have employed in earlier chapters. That is to say, we will assume that the aperture is uniformly illuminated and that the phase $\phi = 0$ across the entire aperture. As we also drop the terms that are quadratic in X and quadratic in Y, equation (6) becomes,

$$\mathbf{A}(x_1, y_1) \approx A_0\, a_0 \int \int \exp\{ -i\,[2\pi/(\lambda L)]\,[\,x_1 X + y_1 Y \cos\psi\,]\}\, dX\, dY\,.$$

$$= A_0\, a_0 \int \exp\{ -i\,\nu X\}\, dX \int \exp\{ -i\,\mu Y\}\, dY\,.$$

$$= A_0\, a_0\, (w\,h)\, e^{[\,-i\,(\Theta_1 + \vartheta_1)\,]}\, \mathrm{sinc}(\alpha_1)\, \mathrm{sinc}(\beta_1)\,,$$

[Equation 18]

where,

$$\nu \equiv [2\pi x_1/(\lambda L)]\,,$$

$$\mu \equiv [2\pi y_1/(\lambda L)]\,\cos\psi\,,$$

$$\alpha_1 \equiv \nu\,[\,X_1 - X_2\,]/2 = \nu\,h/2\,,$$

$$\Theta_1 \equiv \nu\,[\,X_1 + X_2\,]/2\,,$$

$$\beta_1 \equiv \mu\,[\,Y_1 - Y_2\,]/2 = \mu\,w/2\,,$$

$$\vartheta_1 \equiv \mu\,[\,Y_1 + Y_2\,]/2\,.$$

[Equation 19]

In the spirit of the discussion at the end of [Chapter II.A](), we also note that this result may be presented in the following general form:

$$\mathbf{A} (x_a, y_a, z_a) \approx A_0 \, a_0 \, (w \, h) \, e^{[\, -i \, (\Theta_1 + \vartheta_1)\,]} \, \text{sinc}(\, \alpha_1 \,) \, \text{sinc}(\, \beta_1 \,) \, ,$$

[Equation 20]

where,

$$\alpha_1 = k_\alpha \, x_a,$$
$$\Theta_1 = k_\Theta \, x_a,$$
$$\beta_1 = k_\beta \, y_a,$$
$$\vartheta_1 = k_\vartheta \, y_a,$$

$$k_\alpha \equiv \pi \, (\, \lambda \, L \,)^{-1} \, [\, X_1 - X_2 \,] \, ,$$
$$k_\Theta \equiv \pi \, (\, \lambda \, L \,)^{-1} \, [\, X_1 + X_2 \,] \, ,$$
$$k_\beta \equiv \pi \, (\, \lambda \, L \,)^{-1} \, [\, Y_1 - Y_2 \,] \, ,$$
$$k_\vartheta \equiv \pi \, (\, \lambda \, L \,)^{-1} \, [\, Y_1 + Y_2 \,] \, ,$$

and,

$$L = [\, z_a^2 + y_a^2 + x_a^2 \,]^{1/2}.$$

[Equation 21]

Then we simply note that for any point (x_1, y_1) on a flat image plane that is tilted with respect to the rectangular aperture as illustrated in Figure II.2, $x_a = x_1$, $y_a = (\, y_1 \cos\psi \,)$, and $z_a = (\, Z - y_1 \sin\psi \,)$.

Footnotes

[1]Initially, it seemed a bit odd to me that Leseberg & Frère would prescribe $\phi(x', y')$ in this particular way. This phase function makes sense if the light that is illuminating the rectangular surface (the aperture) originates as a coherent source *from* the image screen; but then one would not expect the light to bounce off of the surface and head directly back to the image screen. Eventually I realized that Leseberg & Frère picked up this particular prescription for ϕ from the article by Patorski (1983). As an experimentalist, Patorski was pointing his coherent source of light directly at the image screen, then inserting a tilted, *transparent* aperture between the light source and the screen. Hence, the object [aperture] was being "illuminated by an on-axis plane wave."

[2] If a_0 is the brightness *per unit area* of the coherent plane wave whose propagation vector is perpendicular to the image screen, as assumed by Patorski (1983), then when the tilted transparent

aperture is inserted between the light source and the screen, the brightness per unit area *on* the aperture will be $u_0(x,y) = a_0 \cos\psi$. In conjunction with equation (8), therefore, the complex amplitude of light that is incident (and, in this case, also leaving) the aperture will be,

$$I(x,y) = a_0 \cos\psi \exp\{ i (2\pi/\lambda) y' \sin\psi \} .$$

[3]In order to see the correspondence between our equation (12) and Patorski's equation (2), note that Patorski uses the angle β instead of ψ, and that the ratio $(z_0/r_0) = [1 - (p^2 + q^2)]^{1/2}$. Also, there are two typesetting mistakes in Patorski's equation (2): in one place the coefficient "g" appears instead of a "q", and a factor of y' has been inadvertently omitted from the last term inside the exponential.

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II.C Rectangular Aperture with an Arbitrary (Euler Angle) Orientation
with Respect to the Hologram

Here we wish to consider the situation where the rectangular aperture (2D surface of a solid rectangle) has an arbitrary orientation with respect to the image screen (hologram), and derive the transformation equations relating the (x_h, y_h) coordinates of any point on the surface of the hologram to the Cartesian coordinate system (x_a, y_a, z_a) that is affixed to the aperture. Once the three functional expressions $x_a(x_h, y_h)$, $y_a(x_h, y_h)$, and $z_a(x_h, y_h)$ have been derived, the expressions presented at the end of § II.B, namely,

$$\mathbf{A}\,(x_a, y_a, z_a) \approx A_0\, a_0\, (w\, h)\, e^{[\,-\,i\,(\Theta_1 + \vartheta_1)\,]}\, \text{sinc}(\alpha_1)\, \text{sinc}(\beta_1)\,,$$

[Equation 1]

where,

$$\alpha_1 = k_\alpha\, x_a,$$
$$\Theta_1 = k_\Theta\, x_a,$$
$$\beta_1 = k_\beta\, y_a,$$
$$\vartheta_1 = k_\vartheta\, y_a,$$

$$k_\alpha \equiv \pi\,(\lambda L)^{-1}\,[\,X_1 - X_2\,]\,,$$
$$k_\Theta \equiv \pi\,(\lambda L)^{-1}\,[\,X_1 + X_2\,]\,,$$
$$k_\beta \equiv \pi\,(\lambda L)^{-1}\,[\,Y_1 - Y_2\,]\,,$$
$$k_\vartheta \equiv \pi\,(\lambda L)^{-1}\,[\,Y_1 + Y_2\,]\,,$$

and,

$$L = [\, z_a^2 + y_a^2 + x_a^2\,]^{1/2}.$$

[Equation 2]

can be immediately used to determine the complex amplitude of light at all positions on the hologram that will result from one rectangular aperture.

General Coordinate Transformations

It is customary to define the relative orientation of two Cartesian coordinate systems in terms of two Euler angles (γ, δ), as illustrated here in Figure II.2. Note that the Euler angles are introduced such that the 2nd (*i.e.,* blue) coordinate system is "twisted" by an angle γ about the z_1-axis, first, then the 2nd coordinate system is tipped by an angle δ about the x_2-axis. Hence, the x_2 axis lies in the x_1-y_1 plane and defines the line of intersection between the x_1-y_1 and x_2-y_2 planes.


Figure II.2
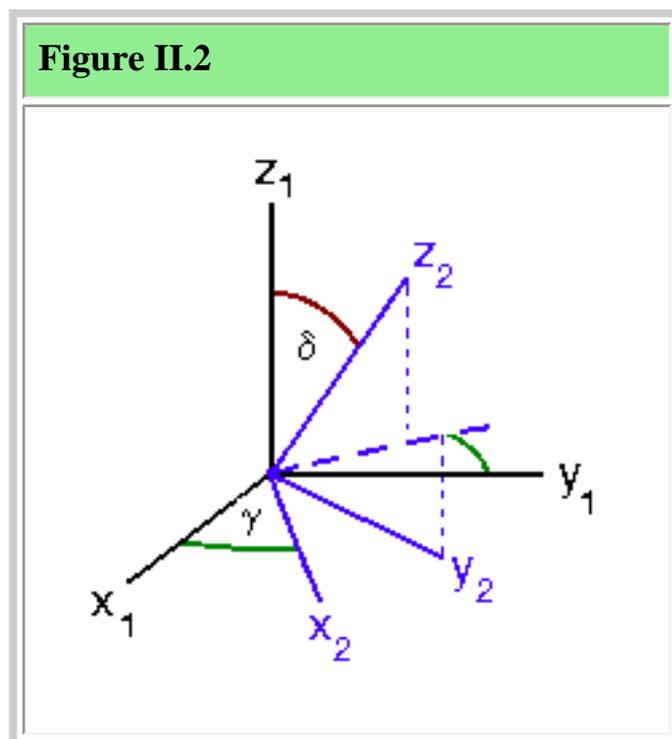
Then, quite generally, the following relationships hold:

Transformation Equations	
2 --> 1	1 --> 2
$x_1 = x_2 \cos\gamma - (y_2 \cos\delta + z_2 \sin\delta) \sin\gamma$	$x_2 = x_1 \cos\gamma + y_1 \sin\gamma$
$y_1 = x_2 \sin\gamma + (y_2 \cos\delta + z_2 \sin\delta) \cos\gamma$	$y_2 = (y_1 \cos\gamma - x_1 \sin\gamma) \cos\delta - z_1 \sin\delta$
$z_1 = z_2 \cos\delta - y_2 \sin\delta \, z_2 \cos\delta$	$z_2 = (y_1 \cos\gamma - x_1 \sin\gamma) \sin\delta + z_1 \cos\delta$

We will also want to consider situations in which the origins of the two coordinate systems do not

coincide, but are offset as illustrated in Figure II.3 by the coordinates (X_0, Y_0, Z_0) as measured in the 1st coordinate system.

Figure II.3
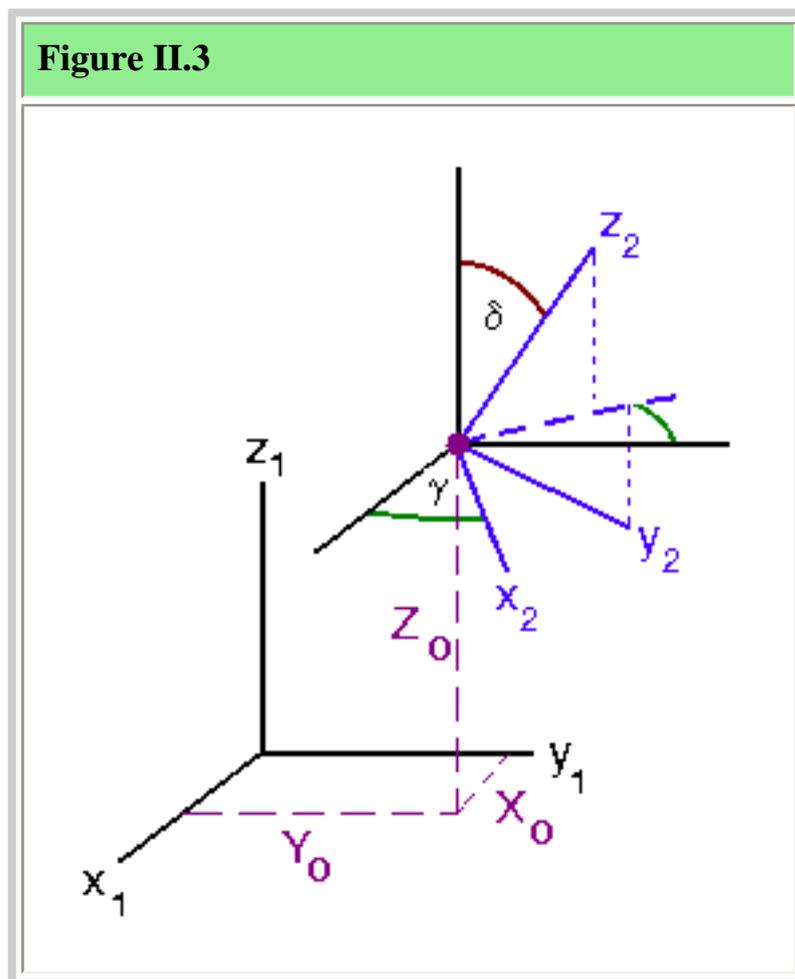
Then the expressions that transform from one system of Cartesian coordinates to the other become,

Transformation Equations	
2 --> 1	1 --> 2
$x_1 = X_0 + x_2 \cos\gamma - (y_2 \cos\delta + z_2 \sin\delta) \sin\gamma$	$x_2 = (x_1 - X_0) \cos\gamma + (y_1 - Y_0) \sin\gamma$
$y_1 = Y_0 + x_2 \sin\gamma + (y_2 \cos\delta + z_2 \sin\delta) \cos\gamma$	$y_2 = [(y_1 - Y_0) \cos\gamma - (x_1 - X_0) \sin\gamma] \cos\delta$ $- (z_1 - Z_0) \sin\delta$
$z_1 = Z_0 + z_2 \cos\delta - y_2 \sin\delta$	$z_2 = [(y_1 - Y_0) \cos\gamma - (x_1 - X_0) \sin\gamma] \sin\delta$ $+ (z_1 - Z_0) \cos\delta$

Specific Cases

$$\boxed{X_0 = Y_0 = 0 \text{ and } \gamma = 0}$$

In §§ II.A and II.B (see especially Figure II.2), we examined the situation where, in the notation of Figure II.3, the 1st coordinate system was affixed to the aperture, the 2nd coordinate system was affixed to the image screen (hologram), and the orientation of the rectangular aperture relative to the screen was such that,

$$\gamma = 0 ,$$
$$\delta = \psi ,$$
$$X_0 = 0 ,$$
$$Y_0 = 0 ,$$
$$Z_0 = Z .$$

By the transformation equations given above, we therefore deduce that every point across the plane of the hologram (for which, by definition, $z_h = 0$), has a position in the reference frame of the aperture given by the coordinates,

$$
\begin{aligned}
x_a \; &= \; X_0 + x_h \cos\gamma - (y_h \cos\psi + z_h \sin\psi) \sin\gamma \\[2mm]
&= \; x_h , \\[2mm]
y_a \; &= \; Y_0 + x_h \sin\gamma + (y_h \cos\psi + z_h \sin\psi) \cos\gamma \\[2mm]
&= \; y_h \cos\psi , \\[2mm]
z_a \; &= \; Z_0 + z_h \cos\psi - y_h \sin\psi \\[2mm]
&= \; Z - y_h \sin\psi .
\end{aligned}
$$

[Equation 3]

This is in complete agreement with the coordinate expressions given in the sentence following equation (21) of § II.B.

$$\boxed{X_0 = Y_0 = 0}$$

Now the results of § II.B can be generalized to the situation where the aperture has an arbitrary (γ, ψ) orientation with respect to the image screen (hologram). By setting $X_0 = Y_0 = 0$, but not forcing the "twist" angle γ to be zero, the transformation equations give,

$$x_a = x_h \cos\gamma - y_h \cos\psi \sin\gamma \,,$$

$$y_a = x_h \sin\gamma + y_h \cos\psi \cos\gamma \,,$$

$$z_a = Z_0 - y_h \sin\psi \,.$$

[Equation 4]

When plugged into equations (1) and (2), above, these coordinates provide the generalized expression for **A** that was alluded to by Patorski (1983) and should produce an "image screen" pattern that closely matches his experimentally produced image in the last figure of his article.

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III.A Constructing Surfaces from VRML Files

General Concept

What I would like to do now is combine my newly gained understanding of how CGH works with my understanding of how the surfaces of 3D objects have traditionally been constucted using voxel-based or wire-frame modeling and ray-tracing techniques to create an extremely efficient means of generating digital holograms of arbitrarily complex surfaces. The first thing to realize along these lines is that most modern 3D-rendering tools expect to receive input in the form of long lists of vertices and polygons. This is, for example, how the virtual-reality markup language (VRML) works. I would like to take advantage of this standard input format and design a CGH algorithm that can compute the holographic image of any surface that is defined by a group of variously connected polygons. With this in mind, [§ B](#) of this chapter describes how the vertices and polygons of an object's surface are usually specified within a VRML file.

I will also take note of the fact that VRML files *usually* contain only the simplest form of polygons, namely, triangles. That is to say, I will expect that my complex surface will be broken down into variously connected *triangles*. Building on the results of earlier chapters, then, in [§ C](#) of this chapter we derive an analytical expression that specifies the holographic pattern of an arbitrarily shaped triangular aperture. The derived expression is expressed entirely in terms of the intrinsic properties of the triangle except that its final expression depends on the coordinates of the single point (x_a, y_a, z_a) in space from which the triangle is being viewed. The coordinates of this viewing point will be expressed in terms of a Cartesian coordinate system that is affixed to the triangular aperture in such a way that the unit vector $\mathbf{k}_a$ is normal to the surface of the triangle and points away from the lighted surface of the triangle; one of the edges of the triangle shall point in the $\mathbf{j}_a$ direction; and the "origin" of this coordinate system is chosen so as to minimize the calculations required in the determination of the hologram.

Finally, in [§ D](#) of this chapter [not yet written!] we combine the results of §§ B & C and demonstrate how to transform from the **object coordinate system** that is used in VRML to specify the location and orientation of each (triangular) polygon on the surface of an object to a coordinate system that is affixed to the triangular aperture.

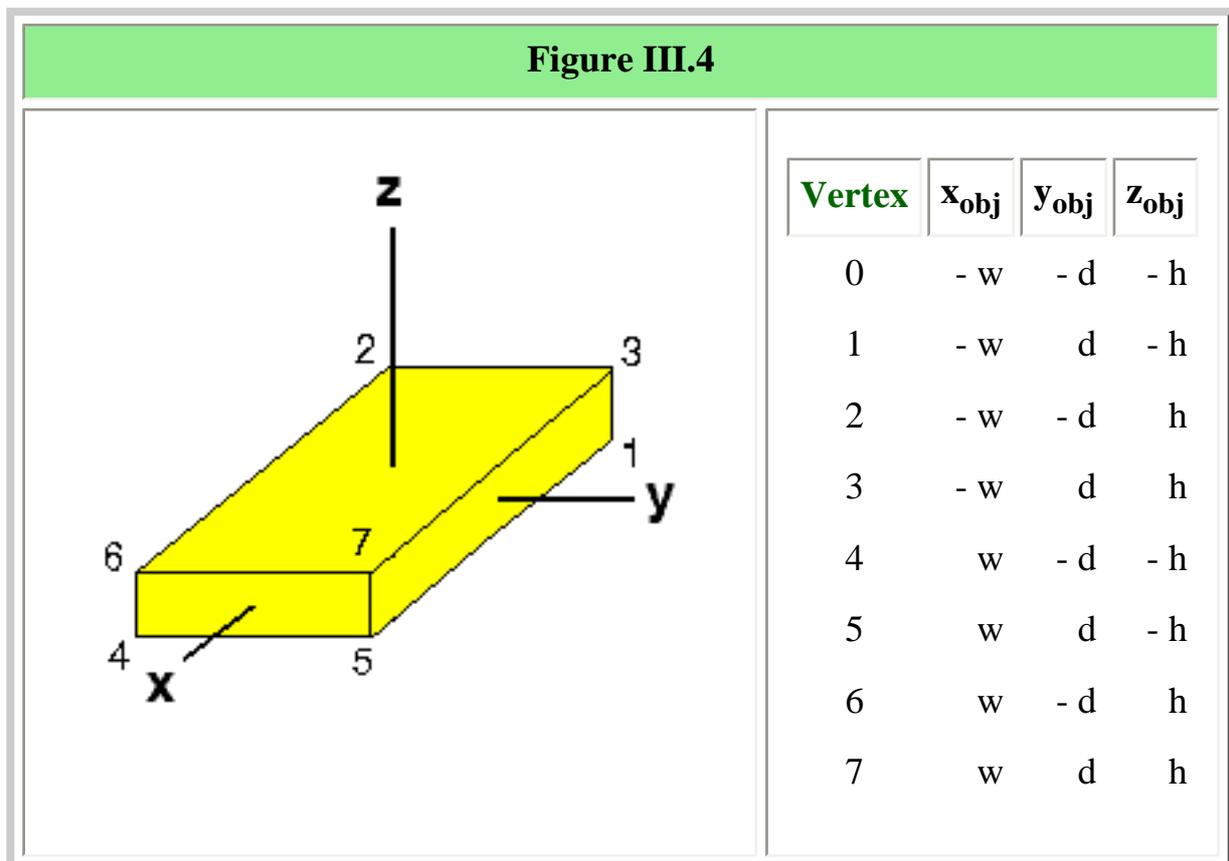| Tohline's Home Page | PHYS 2411 | PHYS 4412 |
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III.B Understanding How Object Surfaces are Specified in VRML Files

In order to define the surface of a three-dimensional object using VRML techniques, one must first define the surface in terms of a "wireframe" model. A wireframe model is constructed by first specifying the coordinates of various strategically placed points (vertices) that lie on the surface of the object, then specifying how subsets of these points are to be connected in sequence in order to form various polygons that coincide with individual (flat) segments of the surface.

Example #1: A Rectangular Parallelepiped

One of the simplest examples is the wireframe model of a rectangular parallelepiped of width 2w, height 2h, and depth 2d, as depicted here in Figure III.4.

Figure III.4			
	Vertex	x_{obj}	y_{obj}	z_{obj}
	0	- w	- d	- h
	1	- w	d	- h
	2	- w	- d	h
	3	- w	d	h
	4	w	- d	- h
	5	w	d	- h
	6	w	- d	h
	7	w	d	h

The entire surface of the rectangular parallelepiped shown in Figure III.4 can be completely specified by eight vertices (numbered 0 through 7 in the figure; but note that the vertex numbered zero is hidden from view) and six four-sided (rectangular) polygons. If we locate the origin of the **object coordinate system** at the geometric center of the rectangular parallelepiped and orient it such that the principal axes of the **object coordinate system** pass through the center of each of the six faces of the object (as depicted in Figure III.4), then the eight vertices will have the coordinate locations identified by the tabulated values in Figure III.4.

At the bottom of this page we have attached the ASCII text of a "wrl" (VRML) file that properly constructs a rectangular parallelepiped very similar to the one shown here (but of a different color). Notice that this file contains the following twelve lines of text:

```
coord Coordinate {
   point [
   -5.0 -2.0 -0.75,
   -5.0  2.0 -0.75,
   -5.0 -2.0  0.75,
   -5.0  2.0  0.75,
    5.0 -2.0 -0.75,
    5.0  2.0 -0.75,
    5.0 -2.0  0.75,
    5.0  2.0  0.75,
   ]
}
```

Each of the eight lines of numbers that lies between the "point" brackets specifies the $(x_{obj}, y_{obj}, z_{obj})$ coordinates in the **object coordinate system** of one of the eight vertices. By design in any wrl file, the first set of coordinates is assigned to vertex number 0; the second set of coordinates is assigned to vertex number 1; and, in general, the nth set of coordinates is assigned to vertex number $(n - 1)$. By comparing the vertex (*i.e.,* point) coordinates in this specific wrl file with the table of vertex coordinates shown in Figure III.4, it is clear that the VRML object that will be drawn from this wrl file will have a half-width w = 5.0, a half-depth d = 2.0, and a half-height h = 0.75.

Point of Clarification

In constructing a wrl file, you need not list the vertices (points) in any particular order. For purposes of clarity only, the order in which they appear in the wrl file printed below has been designed to correspond with the vertex numbering shown in Figure III.4. But once the vertices have been specified in a particular order, *that ordering becomes important* as you move to the next step, which involves specifying which vertices will be connected in sequence in order to define each surface polygon.

The wrl file at the bottom of this page also contains the following eight lines of text:

```
coordIndex [
    0, 2, 3, 1, -1,
    0, 1, 5, 4, -1,
    5, 1, 3, 7, -1,
    5, 7, 6, 4, -1,
    6, 2, 0, 4, -1,
    2, 6, 7, 3, -1,
]
```

Each of the six lines of integers that lies between the "coordIndex" brackets (and is delimited by the integer "-1") specifies which vertices (as numbered in Figure III.4) are to be connected -- and in what order they are to be connected -- in order to construct one of the six faces of the rectangular parallelepiped. For example, the 5th line which reads,

```
    5, 7, 6, 4, -1,
```

instructs your VRML viewer to connect the vertices numbered 5, 7, 6, 4, then back to 5, in a counterclockwise sequence in order to form the front face of the rectangular parallelepiped as illustrated here in Figure III.4. (Note that if you list the same set of vertices but specify that they be connected in a clockwise sequence, your VRML browser will probably make that face of the rectangular parallelepiped transparent and, therefore, it will be invisible.)

Example #2: What is This Object?

In order to test whether or not you understand how to interpret the coordinate date in a VRML file, examine the following lines of text which specify the vertices and polygons of a familiar geometric solid that is not a rectangular parallelepiped.

```
coord Coordinate {
    point [
-3.0  3.0 -4.0,
 3.0  3.0 -4.0,
 0.0 -3.0 -4.0,
 0.0  0.0  4.0,
    ]
}
coordIndex [
    0, 1, 2, -1,
    0, 3, 1, -1,
```

```
    1, 3, 2, -1,
    2, 3, 0, -1,
  ]
```

Now click here in order to download this "mystery" object into your VRML viewer to see whether or not your interpretation is correct.

RECTANGULAR PARALLELEPIPED

The ASCII text that follows forms a complete "wrl" (VRML) file that constructs a rectangular parallelepiped very similar to the one shown above in Figure III.4, although of a different color. If you would like to download this wrl file into your browser to see the rectangular parallelepiped in three dimensions click here. But please **DON'T DOWNLOAD** this "wrl" file unless you are sure you have a working VRML plug-in for your computer.

```
#VRML V2.0 utf8
#Created using createVRML.c

Background {
        skyColor 0.58824 0.24706 0.63922 # purple
}

Transform {
        translation 0 0 0
        rotation 0 0 0 0
        scale 1 1 1
        children DEF Default Viewpoint {
                position 0 0 40
                orientation 0 0 1 0
        }
}

DEF URAMask Transform {
        translation 0 0 0
        rotation 1 0 0 -1.57
        scale 1 1 1
        children [
                Shape {
                        appearance Appearance {
                                material Material { }
                        }
                        geometry IndexedFaceSet {
```

```
                                    creaseAngle 6.283
   coord Coordinate {
      point [
      -5.0  -2.0  -0.75,
      -5.0   2.0  -0.75,
      -5.0  -2.0   0.75,
      -5.0   2.0   0.75,
       5.0  -2.0  -0.75,
       5.0   2.0  -0.75,
       5.0  -2.0   0.75,
       5.0   2.0   0.75,
      ]
   }
   coordIndex [
      0, 2, 3, 1, -1,
      0, 1, 5, 4, -1,
      5, 1, 3, 7, -1,
      5, 7, 6, 4, -1,
      6, 2, 0, 4, -1,
      2, 6, 7, 3, -1,
   ]
                                    color Color {
                                           color [
                  0.087157 0 0.996195,
                  0.2841872 0.3225579 0.996195,
                  0.0789911 0.0368341 0.996195,
                  0.0713949 0.0499912 0.996195,
                  0.087157 4.62558e-07 0.996195,
                  0. 0. 1.0,
                  0. 0.5 1.0,
                                                 ]
                                    }
                                    colorPerVertex TRUE
                  }
                }
         ]
}
```

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III.C Triangular Aperture

Setup

Consider any three points in space -- P_0, P_1, and P_2 -- that do not lie on a straight line. From elementary Euclidian geometry we know that these three points define a plane. They also can be used to specify the vertices of a flat triangular aperture that lies in this plane.

Let's construct a Cartesian coordinate system in which the X-Y plane is coincident with the plane defined by these three points and orient it as illustrated in Figure III.1, that is, such that,

Figure III.1

   a. the Y-axis is parallel to the line segment connecting P_0 and P_2;
   b. a vector pointing from P_0 to P_2 points in the +Y direction;
   c. point P_1 lies to the right of this line; and
   d. the +Z-axis points out of the page.

Now, define the distances X_1, Y_1, and Y_2 such that, if vertex P_0 were coincident with the origin of the Cartesian coordinate system, the coordinates of the three vertices would be:
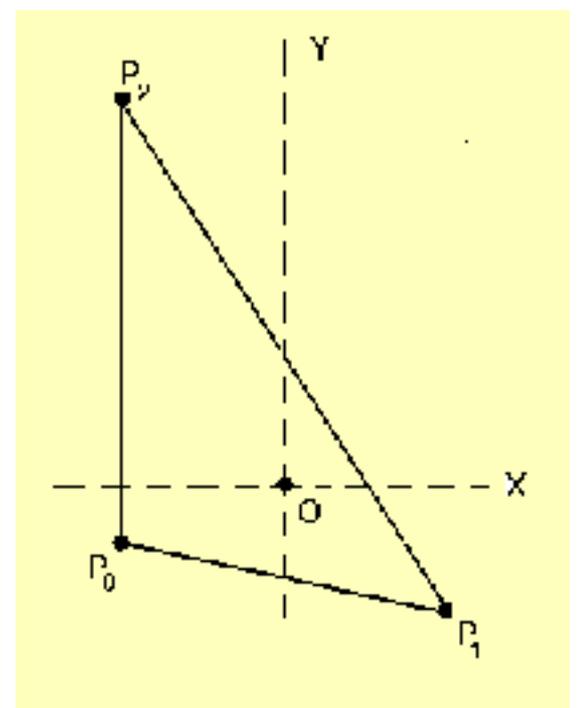
$$P_0 \ = \ (\,0, 0, 0\,)\,,$$

$$P_1 \ = \ (\,X_1, Y_1, 0\,)\,,$$

$$P_2 \ = \ (\,0, Y_2, 0\,)\,.$$

[Equation 1]

(Note that, defined in this way, X_1 and Y_2 are necessarily positive numbers, but Y_1 may be either

positive or negative.)

Finally, shift the Cartesian coordinate system sideways in the X-Y plane so that P_0 is no longer at the origin but is, instead, at the position,

$$P_0 \;=\; [\,-(\,X_1\,/\,2\,)\,,\,-(\,Y_1\,/\,2+Y_2\,/\,4\,)\,,\,0\,]\,,$$

[Equation 2]

as illustrated in Fig. III.3. (The rationale for shifting the origin of the coordinate system to this position will be given below.) As a result, the other two points will be located at positions,

$$P_1 \;=\; [\,(\,X_1\,/\,2\,)\,,\,(\,Y_1\,/\,2-Y_2\,/\,4\,)\,,\,0\,]\,,$$

$$P_2 \;=\; [\,-(\,X_1\,/\,2\,)\,,\,(\,3\,Y_2\,/\,4-Y_1\,/\,2\,)\,,\,0\,]\,.$$

[Equation 3]

We will henceforth refer to the Cartesian coordinate system defined in this way as the **aperture coordinate system**. In the aperture coordinate system, the equation for the line that connects vertices P_2 and P_1, and thereby forms the "upper" edge of the triangular aperture is,

$$Y = [\,(\,Y_1-Y_2\,)\,/\,X_1\,]\,X+Y_2\,/\,4\,,$$

[Equation 4]

and the equation for the line that connects vertices P_0 and P_1, and thereby forms the "lower" edge of the triangular aperture is,

$$Y = [\,Y_1\,/\,X_1\,]\,X-Y_2\,/\,4\,.$$

[Equation 5]

Analytical Result

By analogy with earlier chapters, if the triangular aperture is assumed to be uniformly bright (*i.e.*, $a_{j,k} = a_0\,dX\,dY$, and a_0 is the brightness per unit area), and the phase $\phi = 0$ at all locations on the aperture, then at any position in space (X_a,Y_a,Z_a) as measure *in* the aperture coordinate system *from* the aperture origin (as illustrated above in Fig. III.3),

$$\mathbf{A}\,(X_a,Y_a,Z_a) \approx A_0\,a_0 \int \exp\{-i\,[2\pi\,X_a X\,/\,(\lambda L)]\}\,dX \int \exp\{-i\,[2\pi\,Y_a Y\,/\,(\lambda L)]\}\,dY\,,$$

[Equation 6]

where,

$$L \equiv [\ X_a{}^2 + Y_a{}^2 + Z_a{}^2\ ]^{1/2}\ .$$

[Equation 7]

Both of these integrals can be completed in the same fashion as described in earlier chapters for the 1D slit or for the 2D rectangular aperture, but now some care must be taken in establishing the limits of integration because along two of the edges of the triangular aperture, the limiting values of "Y" are now a function of the X-coordinate. However, these functions are readily specified for the "upper" and "lower" edges of the triangle by equations (4) and (5), respectively. Hence, the integral over Y must be performed between the limits Y_{lower} and Y_{upper}, where,

$$Y_{lower} \equiv Y_1\ (\ X\ /\ X_1\ )\ -\ (\ Y_2\ /\ 4\ )\ ,$$
$$Y_{upper} \equiv (\ Y_1\ -\ Y_2\ )\ (\ X\ /\ X_1\ )\ +\ (\ Y_2\ /\ 4\ )\ ,$$

[Equation 8]

and the integral over X must be performed between the limits X_{lower} and X_{upper}, where,

$$X_{lower} \equiv -\ X_1\ /\ 2\ ,$$
$$X_{upper} \equiv +\ X_1\ /\ 2\ .$$

[Equation 9]

Completing the integrals with these limits gives,

$$\mathbf{A}\ (X_a, Y_a, Z_a) \approx i\ [\ A_0\ a_0\ (\ X_2\ Y_3\ /\ 2\ )\ ]\ \{\ [\ \exp(\ -\ i\ \beta\ )\ /\ (\ 2\ \beta\ )\ ]\ \text{sinc}\ \alpha_{upper}$$
$$-\ [\ \exp(\ +\ i\ \beta\ )\ /\ (\ 2\ \beta\ )\ ]\ \text{sinc}\ \alpha_{lower}\ \}\ ,$$

[Equation 10]

where,

$$\alpha_{upper} \equiv \pi\ X_a\ (\ \lambda\ L\ )^{-1}\ [\ X_1 + (\ Y_1\ -\ Y_2\ )\ (\ Y_a\ /\ X_a\ )\ ]\ ,$$
$$\alpha_{lower} \equiv \pi\ X_a\ (\ \lambda\ L\ )^{-1}\ [\ X_1 + Y_1\ (\ Y_a\ /\ X_a\ )\ ]\ ,$$
$$\beta \equiv \pi\ Y_a\ (\ \lambda\ L\ )^{-1}\ [\ Y_2\ /\ 2\ ]\ .$$

[Equation 11]

Discussion

Now, when comparing equation (10) to equation (I.B-7) -- which is the result for a rectangular

aperture -- we see two key differences. First, because α_{upper} is not equal to α_{lower}, we are unable to combine the terms containing β into a sinc function. (See, however, what happens when $Y_a = 0$, as discussed below.) Second, equation (10) does not contain a phase offset factor in either the X or Y coordinate directions analogous to the terms that contain Θ_1 and ϑ_1 in (I.B-7). Terms of this nature do not appear here precisely because of our strategic positioning of the origin of the aperture coordinate system as illustrated in Figure III.1 and defined by equation (2).

The analytical expression (10) formally poses problems when either $X_a = 0$ or $Y_a = 0$ ($\beta = 0$) because of division by zero. However, equation (10) still provides the correct result if we realize that, when $X_a = 0$,

$$\alpha_{upper} \longrightarrow \pi Y_a (\lambda L)^{-1} [Y_1 - Y_2] ,$$

$$\alpha_{lower} \longrightarrow \pi Y_a (\lambda L)^{-1} [Y_1] .$$

[Equation 12]

When $Y_a = 0$, one can show that equation (10) reduces to the simpler expression,

$$\mathbf{A} (X_a,0,Z_a) \approx A_0 a_0 (X_1 Y_2 / 2) \, sinc [\pi X_a X_1 (\lambda L)^{-1}] .$$

[Equation 13]

Finally we note that, via the general coordinate transformations derived in § II.C, we can now use equation (10), above, to specify the complex amplitude of light at any coordinate location on a holographic screen. When employing these coordinate transformations, care must be taken to define the offset coordinates (X_O,Y_O,Z_O) shown in Figure II.3 in terms of the separation between the origin (usually the center) of the hologram and the particular origin of the aperture coordinate system, as specified above. Then, for example, if $X_O = Y_O = 0$, equation (II.C-4) provides the functions $X_a(x_h,y_h)$, $Y_a(x_h,y_h)$, and $Z_a(x_h,y_h)$, that are required to express the distance L and the angles α_{upper}, α_{lower}, and β entirely in terms of coordinate positions across the hologram.