2 The Fourier Transform

The definition of a one dimensional continuous function, denoted by $f(x)$, the Fourier transform is defined by:

$$F(u) = \int_{-\infty}^{\infty} f(x) \exp(-\imath 2\pi u x)\, \mathrm{d}x \tag{1}$$

with the inverse Fourier transform defined by;

$$f(x) = \int_{-\infty}^{\infty} F(u) \exp(\imath 2\pi u x)\, \mathrm{d}u \tag{2}$$

where it should be noted that the factors of 2π are incorporated into the transform kernel[1].

Some insight to the Fourier transform can be gained by considering the case of the Fourier transform of a *real* signal $f(x)$. In this case the Fourier transform can be separated to give,

$$F(u) = F_r(u) + \imath F_\imath(u) \tag{3}$$

where we have,

$$\begin{aligned} F_r(u) &= \int_{-\infty}^{\infty} f(x) \cos(2\pi u x)\, \mathrm{d}x \\ F_\imath(u) &= -\int_{-\infty}^{\infty} f(x) \sin(2\pi u x)\, \mathrm{d}x \end{aligned}$$

So the real part of the Fourier transform is the decomposition of $f(x)$ in terms of cosine functions, and the imaginary part a decomposition in terms of sine functions. The u variable in the Fourier transform is interpreted as a frequency, for example if $f(x)$ is a sound signal with x measured in seconds then $F(u)$ is its frequency spectrum with u measured in Hertz (s^{-1}).

NOTE: Clearly (ux) *must* be dimensionless, so if x has dimensions of *time* then u *must* have dimensions of $\textit{time}^{-1}$.

This is one of the most common applications for Fourier Transforms where $f(x)$ is a detected signal (for example a sound made by a musical instrument), and the Fourier Transform is used to give the spectral response.

2.1 Properties of the Fourier Transform

The Fourier transform has a range of useful properties, some of which are listed below. In most cases the proof of these properties is simple and can be formulated by use of equation (3) and (4). The proofs of many of these properties are given in the questions and solutions at the back of this booklet.

Linearity: The Fourier transform is a linear operation so that the Fourier transform of the sum of two functions is given by the sum of the individual Fourier transforms. Therefore,

$$\mathcal{F}\{a f(x) + b g(x)\} = a F(u) + b G(u) \tag{4}$$

[1] There are various definitions of the Fourier transform that puts the 2π either inside the kernel or as external scaling factors. The difference between them whether the variable in Fourier space is a "frequency" or "angular frequency". The difference between the definitions are clearly just a scaling factor. The optics and digital Fourier applications the 2π is usually defined to be inside the kernel but in solid state physics and differential equation solution the 2π constant is usually an external scaling factor.

where $F(u)$ and $G(u)$ are the Fourier transforms of $f(x)$ and and $g(x)$ and a and b are constants. This property is central to the use of Fourier transforms when describing *linear* systems.

Complex Conjugate: The Fourier transform of the *Complex Conjugate* of a function is given by

$$\mathcal{F}\{f^*(x)\} = F^*(-u) \tag{5}$$

where $F(u)$ is the Fourier transform of $f(x)$.

Forward and Inverse: We have that

$$\mathcal{F}\{F(u)\} = f(-x) \tag{6}$$

so that if we apply the Fourier transform twice to a function, we get a spatially reversed version of the function. Similarly with the inverse Fourier transform we have that,

$$\mathcal{F}^{-1}\{f(x)\} = F(-u) \tag{7}$$

so that the Fourier and inverse Fourier transforms differ only by a sign.

Differentials: The Fourier transform of the derivative of a functions is given by

$$\mathcal{F}\left\{\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right\} = \imath 2\pi u F(u) \tag{8}$$

and the second derivative is given by

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -(2\pi u)^2 F(u) \tag{9}$$

This property will be used in the DIGITAL IMAGE ANALYSIS and IMAGE PROCESSING I course to form the derivative of an image.

Power Spectrum: The *Power Spectrum* of a signal is defined by the modulus square of the Fourier transform, being $|F(u)|^2$. This can be interpreted as the *power* of the frequency components. Any function and its Fourier transform obey the condition that

$$\int_{-\infty}^{\infty} |f(x)|^2 \,\mathrm{d}x = \int_{-\infty}^{\infty} |F(u)|^2 \,\mathrm{d}u \tag{10}$$

which is frequently known as *Parseval's Theorem*[2]. If $f(x)$ is interpreted at a voltage, then this theorem states that the *power* is the same whether measured in real (time), or Fourier (frequency) space.

2.2 Two Dimensional Fourier Transform

Since the three courses covered by this booklet use two-dimensional scalar potentials or images we will be dealing with two dimensional function. We will define the two dimensional Fourier transform of a continuous function $f(x,y)$ by,

$$F(u,v) = \iint f(x,y) \exp(-\imath 2\pi(ux+vy)) \,\mathrm{d}x\,\mathrm{d}y \tag{11}$$

[2] Strictly speaking Parseval's Theorem applies to the case of Fourier series, and the equivalent theorem for Fourier transforms is correctly, but less commonly, known as Rayleigh's theorem

with the inverse Fourier transform defined by;

$$f(x,y) = \iint F(u,v)\exp(\imath 2\pi(ux+vy))\,\mathrm{d}u\,\mathrm{d}v \tag{12}$$

where the limits of integration are taken from $-\infty \to \infty$[3]

Again for a real two dimensional function $f(x,y)$, the Fourier transform can be considered as the decomposition of a function into its sinusoidal components. If $f(x,y)$ is considered to be an image with the "brightness" of the image at point (x_0,y_0) given by $f(x_0,y_0)$, then variables x,y have the dimensions of length. In Fourier space the variables u,v have therefore the dimensions of *inverse* length, which is interpreted as *Spatial Frequency*.

NOTE: Typically x and y are measured in mm so that u and v have are in units of mm^{-1} also referred to at *lines per mm*.

The Fourier transform can then be taken as being the decomposition of the image into two dimensional sinusoidal *spatial* frequency components. This property will be examined in greater detail the relevant courses.

The properties of one the dimensional Fourier transforms covered in the previous section convert into two dimensions. Clearly the derivatives then become

$$F\left\{\frac{\partial f(x,y)}{\partial x}\right\} = \imath 2\pi u F(u,v) \tag{13}$$

and with

$$F\left\{\frac{\partial f(x,y)}{\partial y}\right\} = \imath 2\pi v F(u,v) \tag{14}$$

yielding the important result that,

$$F\left\{\nabla^2 f(x,y)\right\} = -(2\pi w)^2 F(u,v) \tag{15}$$

where we have that $w^2 = u^2 + v^2$. So that taking the Laplacian of a function in real space is equivalent to multiplying its Fourier transform by a circularly symmetric quadratic of $-4\pi^2 w^2$.

The two dimensional Fourier Transform $F(u,v)$, of a function $f(x,y)$ is a separable operation, and can be written as,

$$F(u,v) = \int P(u,y)\exp(-\imath 2\pi vy)\,\mathrm{d}y \tag{16}$$

where

$$P(u,y) = \int f(x,y)\exp(-\imath 2\pi ux)\,\mathrm{d}x \tag{17}$$

where $P(u,y)$ is the Fourier Transform of $f(x,y)$ with respect to x only. This property of separability will be considered in greater depth with regards to digital images and will lead to an implementation of two dimensional discrete Fourier Transforms in terms of one dimensional Fourier Transforms.

2.3 The Three-Dimensional Fourier Transform

In the three dimensional case we have a function $f(\vec{r})$ where $\vec{r} = (x,y,z)$, then the three-dimensional Fourier Transform

$$F(\vec{s}) = \iiint f(\vec{r})\exp(-\imath 2\pi \vec{r}.\vec{s})\,\mathrm{d}\vec{r}$$

[3]Unless otherwise specified all integral limits will be assumed to be from $-\infty \to \infty$

where $\vec{s} = (u, v, w)$ being the three reciprocal variables each with units length$^{-}1$. Similarly the inverse Fourier Transform is given by

$$f(\vec{r}) = \iiint F(\vec{s}) \exp(\imath 2\pi \vec{r}.\vec{s})\, \mathrm{d}\vec{s}$$

This is used extensively in solid state physics where the three-dimensional Fourier Transform of a crystal structures is usually called "Reciprocal Space"[4].

The three-dimensional Fourier Transform is again separable into one-dimensional Fourier Transform. This property is independent of the dimensionality and one multi-dimensional Fourier Transform can be formulated as a series of one dimensional Fourier Transforms.

[4]This is also referred to as $\vec{k}$-space where $\vec{k} = 2\pi\vec{s}$