Topic 4: Imaging of Extended Objects

Aim: Covers the imaging of extended objects in coherent and incoherent light. The effect of simple defocus is also considered.

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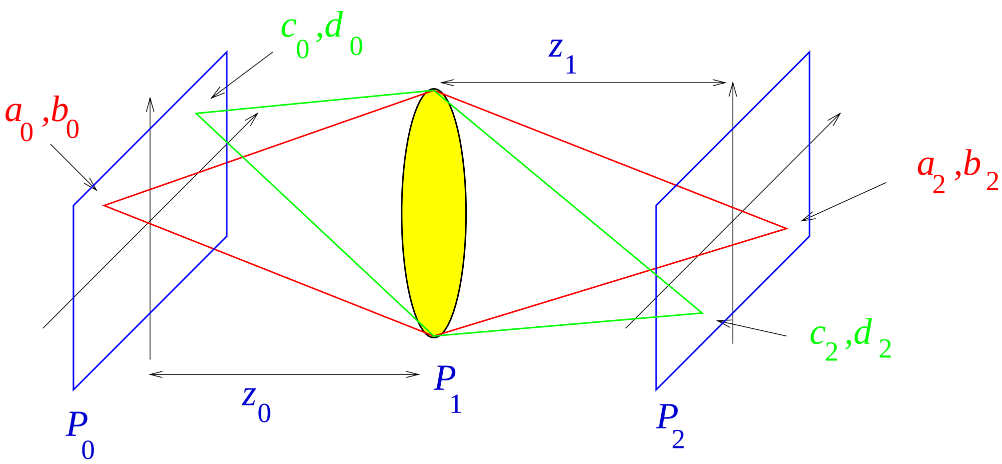
Image of Two Points

If we assume the system is Space Invariant,



Two points sources at (a_0, b_0), and (c_0, d_0) in P_0 we get in P_2 Two PSF located at (a_2, b_2) and (c_2, d_2) where

$$a_2 = -\frac{z_1}{z_0} a_0 \quad ; \quad b_2 = -\frac{z_1}{z_0} b_0$$
$$c_2 = -\frac{z_1}{z_0} c_0 \quad ; \quad d_2 = -\frac{z_1}{z_0} d_0$$

So in P_2 we get amplitude

$$A u_2(x - a_2, y - b_2) \underbrace{\textbf{PLUS}}_{?} B u_2(x - c_2, y - d_2)$$

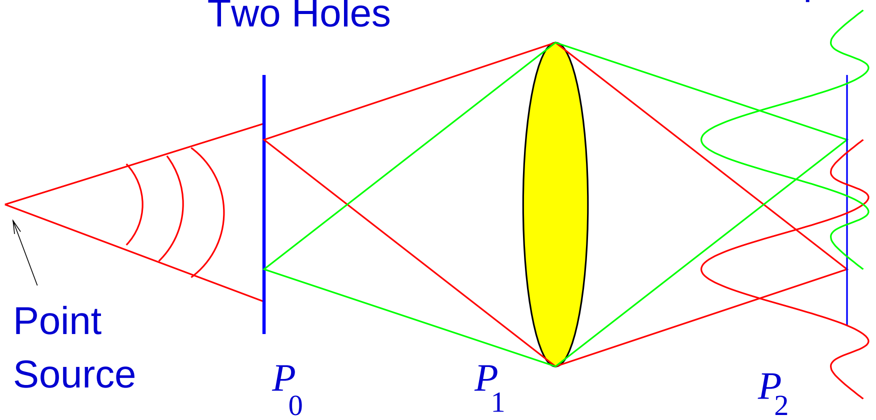
where A and B are the brightness of the points.

What does **PLUS** mean

Depends on the physical properties of the two sources.

Coherent Sources

If the two sources originate from the same source, (eg. Young's Slits), then their amplitudes will sum.



In P_2 Intensity will be,

$$g(x,y) = |Au_2(x-a_2,y-b_2) + Bu_2(x-c_2,y-d_2)|^2$$

These points are said to be **Coherent**

Incoherent Sources

Two point sources completely independent, (2 stars, 2 light bulbs, 2 LED), then their **INTENSITIES** sum.

In P_2 Intensity will be,

$$g(x,y) = |Au_2(x-a_2,y-b_2)|^2 + |Bu_2(x-c_2,y-d_2)|^2$$

These points are said to be **Incoherent**

Coherent and Incoherent are two extremes, the mixture is covered by Partial Coherence. (Not part of this course).

Extended Objects

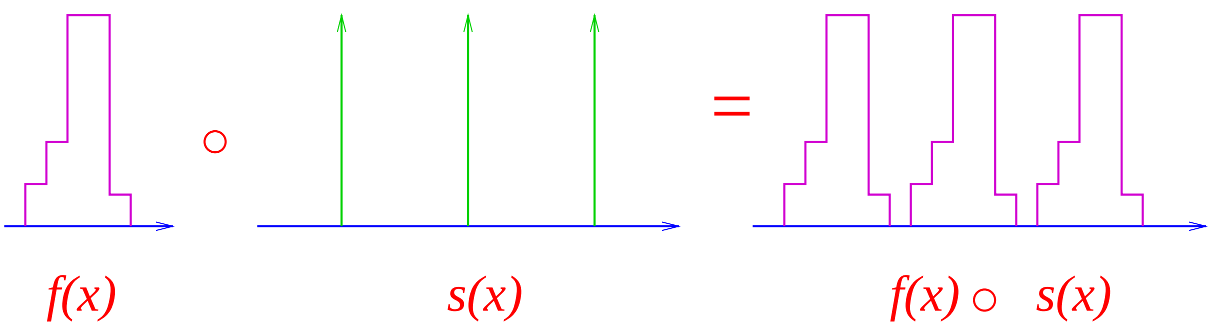
Consider an extended object to be an array of δ-functions.

Picture contains 128^2 points.

Each point of the object is imaged through the optical system and forms PSF.

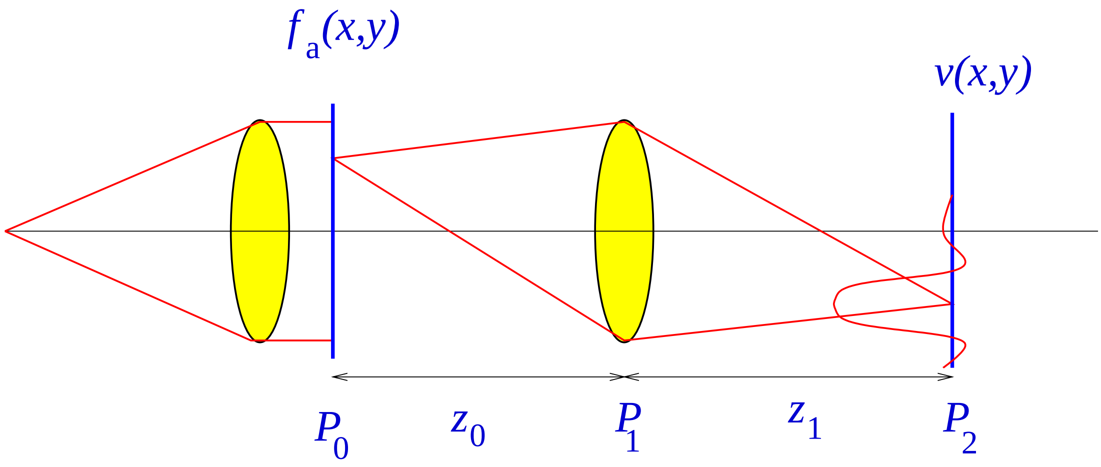
Output image is “Combination” of these PSFs, either Coherently, or Incoherently.

Remember Convolution Relation: (for Fourier Transform Booklet)



Coherent Imaging

All points illuminated from a single point source.



$$\text{Amplitude PSF} = u_2(x,y)$$

Take special case of Unit Magnification, $z_0 = z_1 = 2f$

Also reverse the direction of the coordinates in Plane P_2. So at point (x_0, y_0) we get

$$v(x_0,y_0) = f_a(x_0,y_0) + \sum \text{Parts of other PSFs}$$

so we get that

$$v(x,y) = \int\int f_a(x-s, y-t) \underbrace{u_2(s,t)}_{\text{PSF}} \mathrm{d}s\mathrm{d}t$$

so we have that

$$v(x,y) = f_a(x,y) \odot u_2(x,y)$$

so the intensity distribution in P_2 is given by

$$g(x,y) = |\, f_a(x,y) \odot u_2(x,y)|^2$$

Cont:

Apply the Convolution Theorem, we get that

$$V(u,v) = F(u,v)\,U(u,v)$$

the effect of the lens is to Multiply by the *Filter Function* $U(u,v)$

Define:

$$U(u,v) = \text{Coherent Transfer Function, (CTF)}$$

CTF is the Fourier Transform of the amplitude PSF, so CTF is a scaled version of the Pupil Function.

$$U(u,v) = p(u\lambda z_1, v\lambda z_1)$$

Note on Units, $u\,\&\,v$ have units m^{-1} (Spatial Frequency), while the Pupil function has units of m, (physical size).

Ideal Lens:

$$\begin{aligned} p(x,y) &= 1 \quad x^2 + y^2 \leq a^2 \\ &= 0 \quad \text{else} \end{aligned}$$

so that the CTF will be

$$\begin{aligned} U(u,v) &= 1 \quad u^2 + v^2 \leq w_0^2 \\ &= 0 \quad \text{else} \end{aligned}$$

where we have the Spatial Frequency limit

$$w_0 = \frac{a}{\lambda z_1}$$

to the lens acts like a “Low Pass Filter” with

$$\begin{aligned} \text{Spatial Frequency} &< W_0 \quad \text{passed} \\ \text{Spatial Frequency} &> W_0 \quad \text{blocked} \end{aligned}$$

cont:

For a distant object, we have $z_1 \to f$, so maximum spatial frequency passed by a lens,

$$w_{\max} = \frac{a}{\lambda f}$$

which can be written as

$$w_{\max} = \frac{1}{2\mathrm{F}_{\mathrm{No}}\lambda}$$

so the CTF depends ONLY on the F_{No} of the lens.

Example:

100 mm focal length, $\mathrm{F}_{\mathrm{No}} = 4$ lens (25 mm diameter). for $\lambda = 550$nm

$$w_{\max} = 227 \text{ cycles/mm}$$

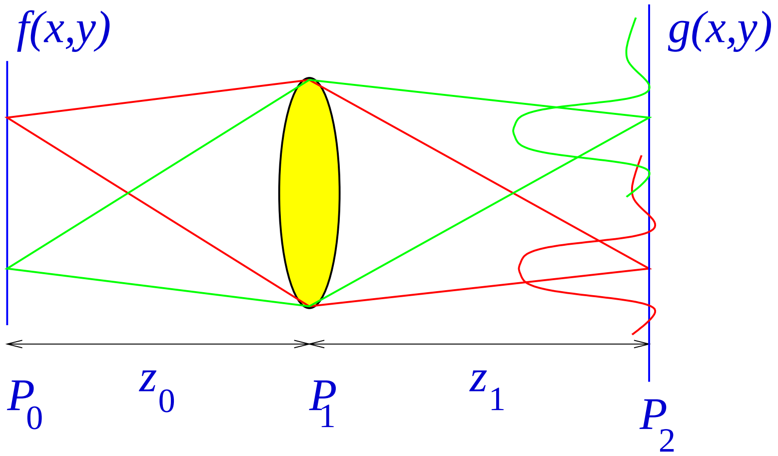
ie

Grating of Frequency	$<$	227 cycles/mm imaged
Grating of Frequency	$>$	227 cycles/mm not imaged

Coherent imaging is investigated in detail in Optical Processing section. (Little more complicated when we include phase).

Incoherent Imaging (Camera)

Assume NO interference between points, reasonable model for photographic images of a natural scene. Input intensity image $f(x,y)$,



The PSF of the system, in incoherent light, is

$$h(x,y) = |u_2(x,y)|^2$$

Imaging as for the coherent case, is that

$$\underbrace{g(x,y)}_{\text{Image}} = \underbrace{f(x,y)}_{\text{Object}} \odot \underbrace{h(x,y)}_{\text{PSF}}$$

So in Fourier space we have that

$$G(u,v) = F(u,v)\,H(u,v)$$

where $H(u,v)$ is known as the "Optical Transfer Function" (OTF).

The OTF acts like a Fourier Space filter and determines the imaging characteristics of the lens.

Cont:

Note that

$$H(u,v) = \mathcal{F}\{h(x,y)\} = \mathcal{F}\{|u_2(x,y)|^2\}$$

so from the Correlation Theorem, we have that

$$H(u,v) = U(u,v) \otimes U(u,v)$$

OTF is Auto-correlation of CTF

So for a lens of pupil function $p(x,y)$ the OTF is given by

$$H(u,v) = p(u\lambda z_1, v\lambda z_1) \otimes p(u\lambda z_1, v\lambda z_1)$$

Again for a distant object, $z_1 \to f$ then the OTF becomes

$$H(u,v) = p(u\lambda f, v\lambda f) \otimes p(u\lambda f, v\lambda f)$$

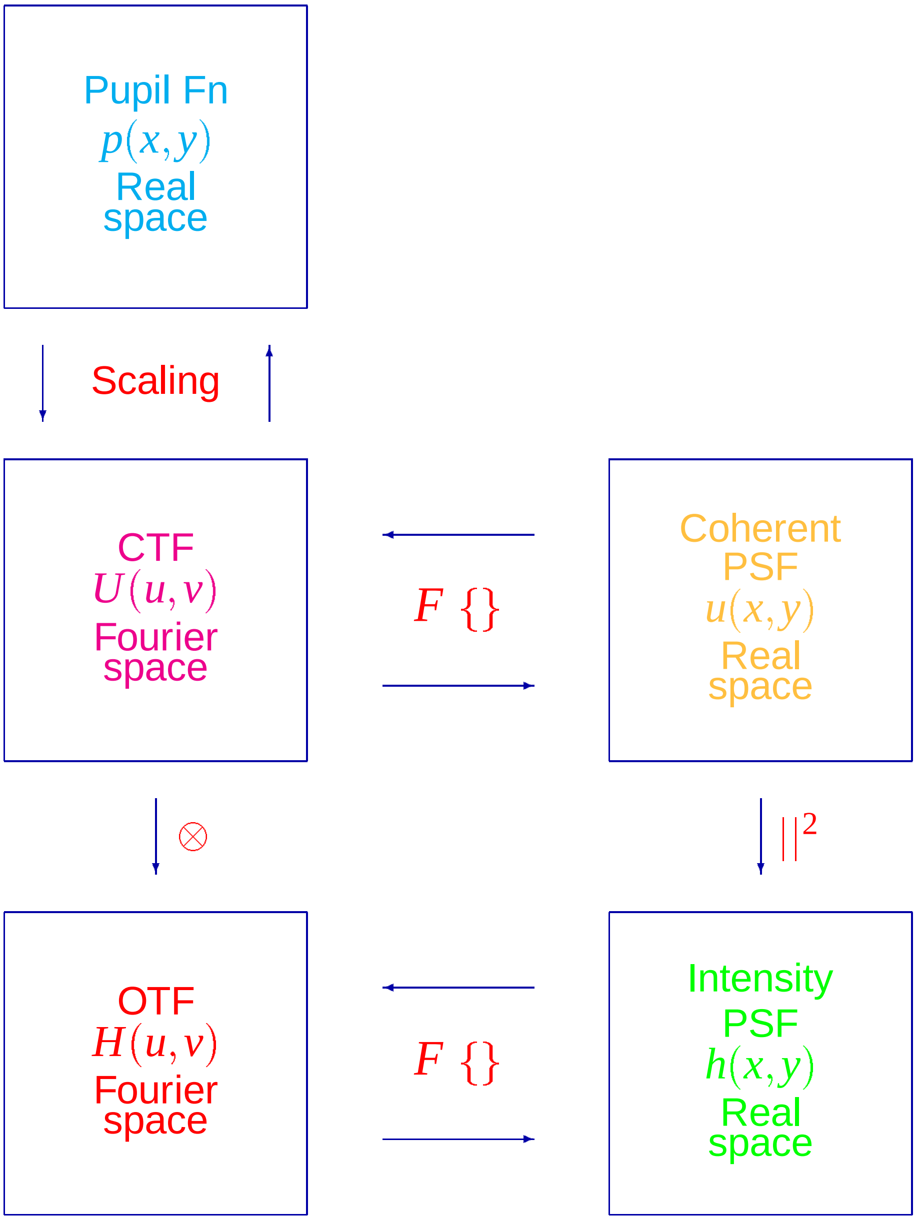
so we can determine the OTF from the Pupil Function of the lens.

Note: This is true for all pupil functions, even if they include aberrations.

Since the OTF is the auto-correlation of the CTF it will be "wider" then the CTF. So optical system will pass higher frequency grating in incoherent light.

Better Resolution in Incoherent Light

Summary of Optical Measures



Note if you know $p(x,y)$ or $u(x,y)$ you can calculate $H(u,v)$ and $h(x,y)$ but NOT VICE-VERSA.

We are not able to determine the properties of a lens (or optical system), in *coherent light* from measures taken in *incoherent light*.

OTF of Round Lens

We have that the OTF is given by

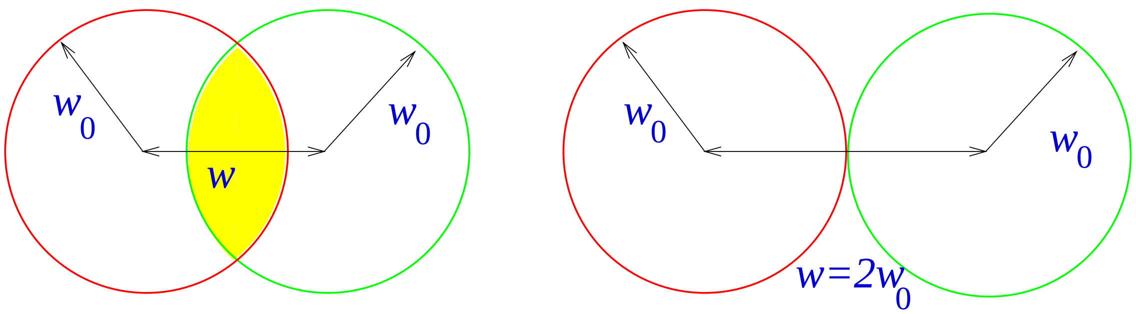
$$H(u,v) = U(u,v) \otimes U(u,v)$$

and for a simple circular lens,

$$\begin{aligned} U(u,v) &= 1 \quad u^2+v^2 \le w_0^2 \\ &= 0 \quad \text{else} \end{aligned}$$

Pictorial Example:

Area of overlap of two shifted circles.



where

$$w^2 = u^2 + v^2$$

So OTF will be circularly symmetric.
Also:

$$H(u,v) = 0 \quad w > 2w_0$$

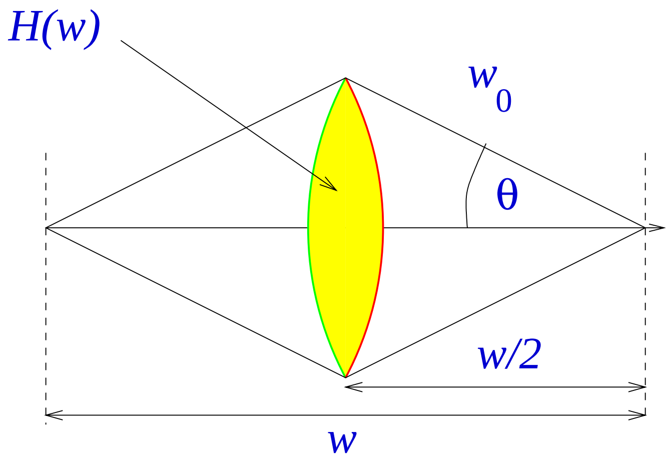
so the frequency limit for incoherent light is,

$$2w_0 = \frac{2a}{\lambda z_1} = \frac{d}{\lambda z_1}$$

This is TWICE the limit for coherent light

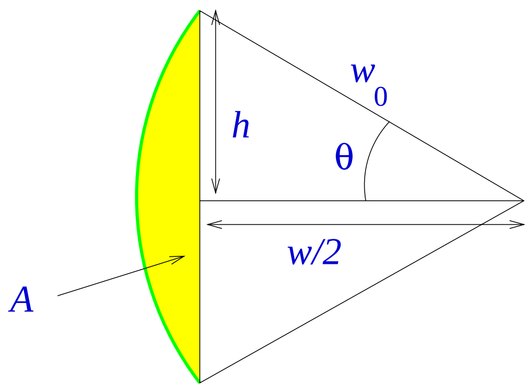
Full Calculation

Look at area of overlap,



where $H(w)$ is overlap of the two circles.

Take half the area,



$$A = \text{Area of Arc} - \text{Area of Triangle}$$

so that

$$A = \frac{2\theta}{2\pi}\pi w_0^2 - \frac{hw}{2} = \theta w_0^2 - \frac{hw}{2}$$

The OFT $H(w)$ is twice this, so that

$$H(w) = 2\theta w_0^2 - hw$$

We have that

$$\cos\theta = \frac{w}{2w_0} \quad \& \quad h = \sqrt{w_0^2 - \frac{w^2}{4}}$$

substitute these into expression for $H(w)$ and we get

$$H(w) = 2w_0^2 \cos^{-1}\left(\frac{w}{2w_0}\right) - ww_0\left(1 - \left(\frac{w}{2w_0}\right)^2\right)^{\frac{1}{2}}$$

it is conventional to normalised so that $H(0) = 1$, so we get

$$H(w) = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{w}{2w_0}\right) - \frac{w}{2w_0}\left(1 - \left(\frac{w}{2w_0}\right)^2\right)^{\frac{1}{2}}\right]$$

or by defining $v_0 = 2w_0$ we have that,

$$H(w) = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{w}{v_0}\right) - \frac{w}{v_0}\left(1 - \left(\frac{w}{v_0}\right)^2\right)^{\frac{1}{2}}\right]$$

where for a image plane distance of z_1,

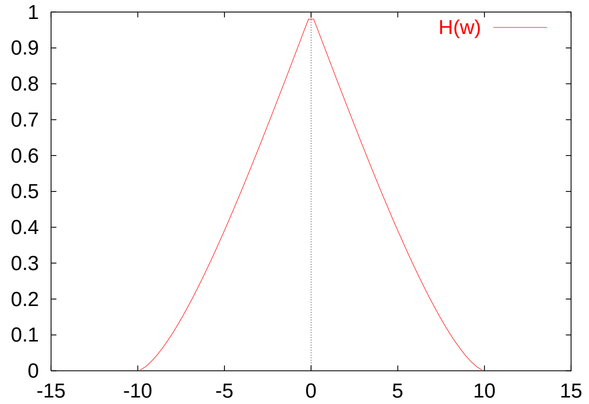
$$v_0 = \frac{2a}{\lambda z_1} = \frac{d}{\lambda z_1}$$

while for a distant object, where $z_1 \to f$

$$v_0 = \frac{2a}{\lambda f} = \frac{d}{\lambda f} = \frac{1}{\lambda \mathrm{F_{No}}}$$

Shape of OTF

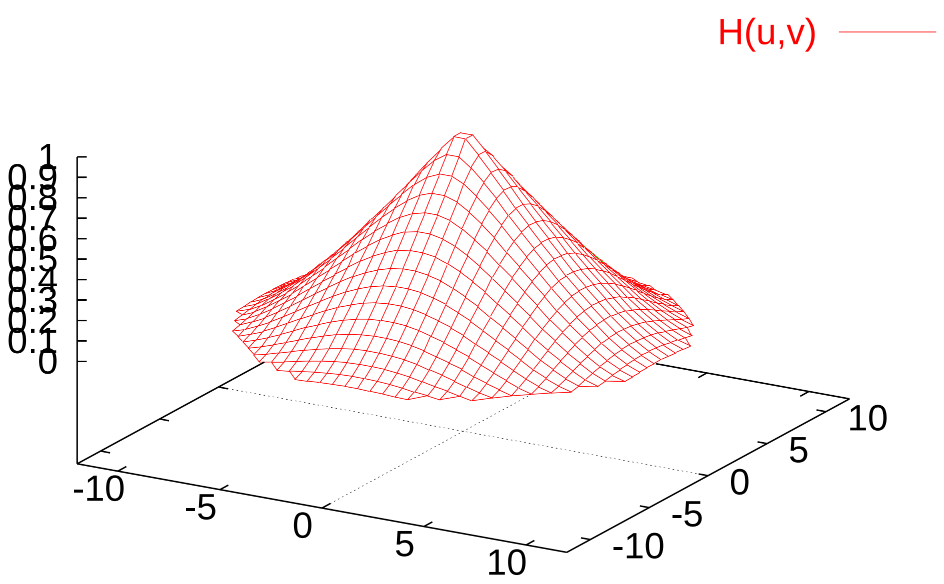
The OTF is "tent" shaped (for $v_0 = 10$),



so spatial frequencies passed up to the limit of v_0, but **NOT** with equal amplitude.

Different than Coherent Case

The OTF is circularly symmetric, so shape is given by



Meaning of OTF

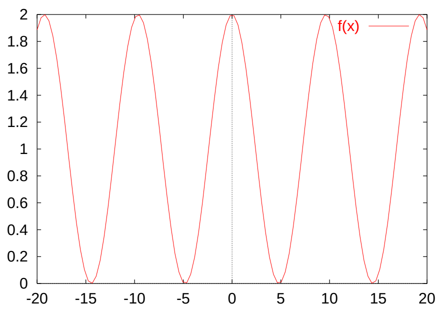
Take object of cosine grating of period $1/a$,

$$f(x,y) = 1 + \cos(2\pi a x)$$

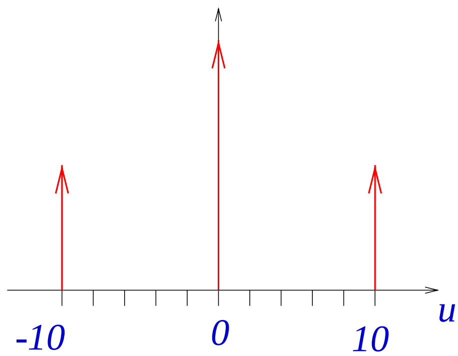
This Fourier Transforms to:

$$F(u,v) = \delta(0) + \frac{1}{2}\delta(u+a) + \frac{1}{2}\delta(u-a)$$

For $a = 10$ we therefore have



Input Function $f(x)$



Fourier Transform $F(u)$

Define: Contrast of object as

$$c = \frac{I_{\max} - I_{min}}{I_{\max} + I_{min}}$$

so contrast of $f(x,y)$ is 1.

Image this grating through lens with OTF $H(w)$, so applying the OTF in Fourier space, we get

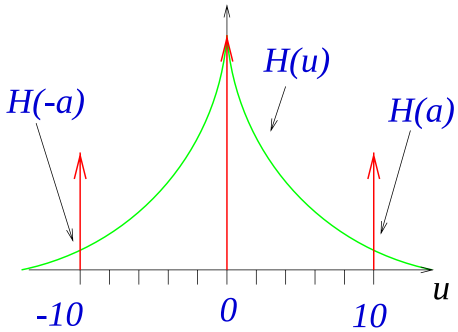
$$\begin{aligned} G(u,v) &= F(u,v)H(u,v) \\ &= \delta(0) + \frac{1}{2}\delta(u+a)H(a) + \frac{1}{2}\delta(u-a)H(a) \end{aligned}$$

where we note that $H(0) = 1$ and $H(a) = H(-a)$.

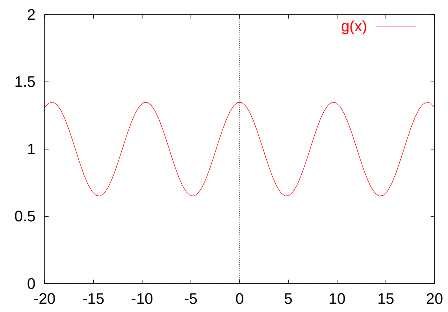
The output is the inverse Fourier Transform, which gives

$$g(x,y) = 1 + H(a)\cos(2\pi a x)$$

which is the same shape as $f(x,y)$, but contrast of $g(x,y)$ is $H(a)$.



Fourier Space $F(u)H(u)$



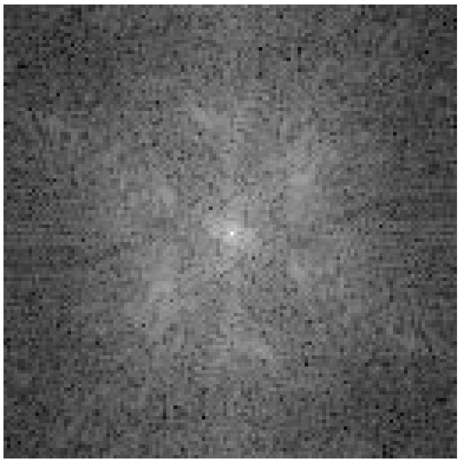
Output $g(x)$

So $H(w)$ is just the contrast with which a grating of spacing $1/w$ is imaged by the optical system.

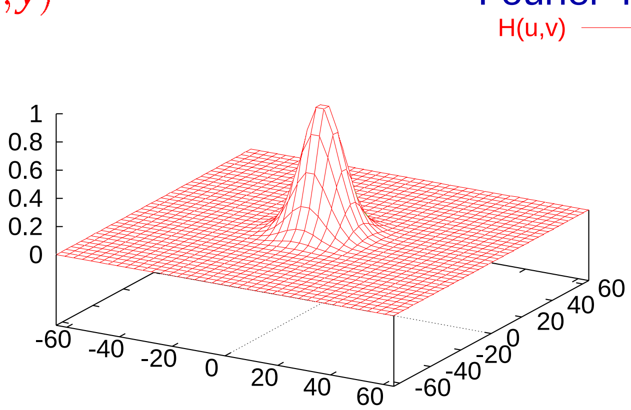
The OTF is a characteristic measure of how well the lens, or optical system will image a particular object.

Digital Image Example

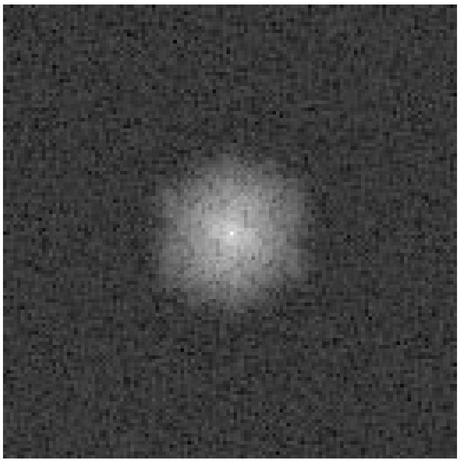
Input image $f(x,y)$



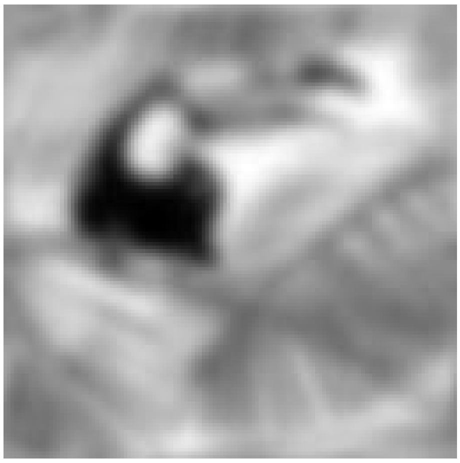
Fourier Transform $F(u,v)$



The OTF (actually Guassian)



Fourier Space $F(u,v)\,H(u,v)$



Output Image $g(x,y)$

CTF and OTF Under Aberrations

In the presence of aberrations the Pupil Function become Complex

$$q(x,y) = p(x,y)\exp(\imath\kappa W(x,y))$$

CTF (Coherent Imaging)

The CTF is the scaled Effective Pupil Function,

$$U(u,v) = q(u\lambda z_1, v\lambda z_1)$$

For Defocus: [Easiest case]

$$W(x,y) = \Delta W \frac{(x^2+y^2)}{a^2}$$

so scaled CTF becomes

$$U(u,v) = \exp\left(\imath\kappa\Delta W \frac{(u^2+v^2)}{w_0^2}\right) \quad \text{for } u^2+v^2 \leq w_0^2$$

where we have that

$$w_0 = \frac{a}{\lambda z_1}$$

$U(u,v)$ is Complex and different Spatial Frequences are phase shifted by different anoumts. No easy solutions.

OTF (Incoherent Imaging)

Again we have that

$$H(u,v) = U(u,v) \otimes U(u,v)$$

Mathematics too difficult for circular aperture, so look at square aperture.

Square Aperture

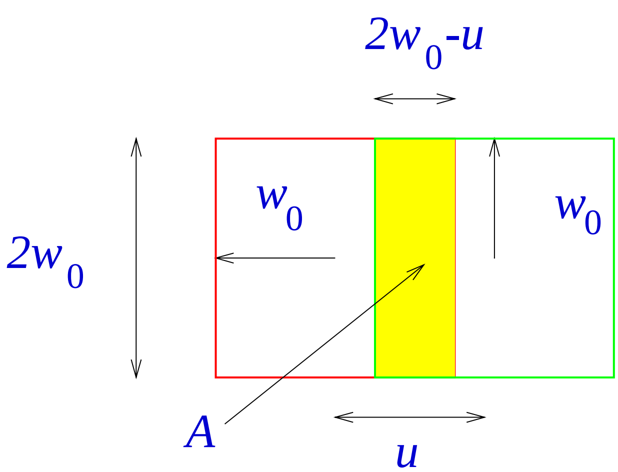
Aperture of size $2a$ by $2a$,

$$\begin{aligned} p(x,y) &= 1 \quad |x| \,\&\, |y| \leq a \\ &= 0 \quad \text{else} \end{aligned}$$

so that the Coherent Transfer Function, CTF is

$$\begin{aligned} U(u,v) &= 1 \quad |u| \,\&\, |v| \leq w_0 \\ &= 0 \quad \text{else} \end{aligned}$$

Again the OTF is given by the Auto-correlation of the CTF



Area of overlap

$$A = (2w_0 - |u|)2w_0$$

cont:

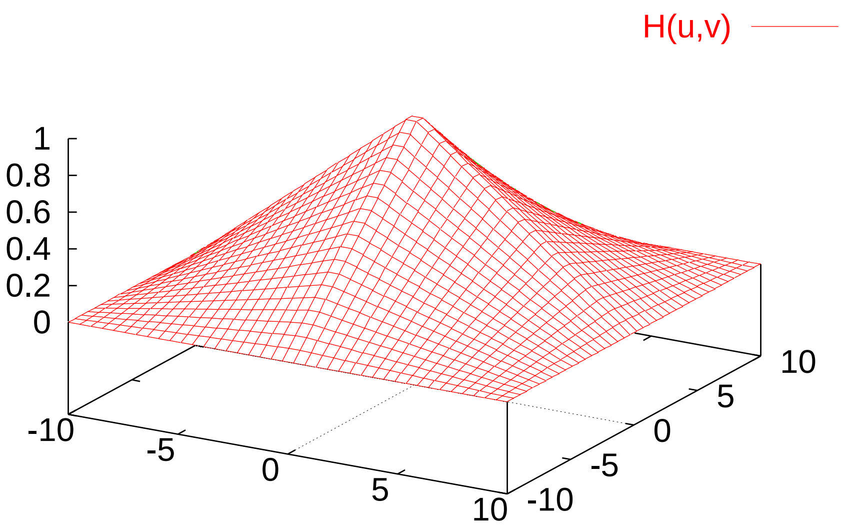
So if we normalise so that $H(0,0) = 1$, then

$$H(u,0) = \left(1 - \frac{|u|}{v_0}\right)$$

where $v_0 = 2w_0$. Then in two dimensions we get

$$H(u,v) = \left(1 - \frac{|u|}{v_0}\right)\left(1 - \frac{|v|}{v_0}\right)$$

which has the same basic shape as for a round lens,



So we expect that results for the square aperture will be very similar to the circular aperture.

Defocus with Square Aperture

Pupil function is

$$\begin{aligned} p(x,y) &= \exp\left(\imath\kappa\Delta W\left(\frac{x^2+y^2}{a^2}\right)\right) \quad |x|\ \&\ |y| \leq a \\ &= 0 \quad \text{else} \end{aligned}$$

so CTF is the same shape, but scaled. (same value at edge)

$$\begin{aligned} U(u,v) &= \exp\left(\imath\kappa\Delta W\left(\frac{u^2+v^2}{w_0^2}\right)\right) \quad |u|\ \&\ |v| \leq w_0 \\ &= 0 \quad \text{else} \end{aligned}$$

Define

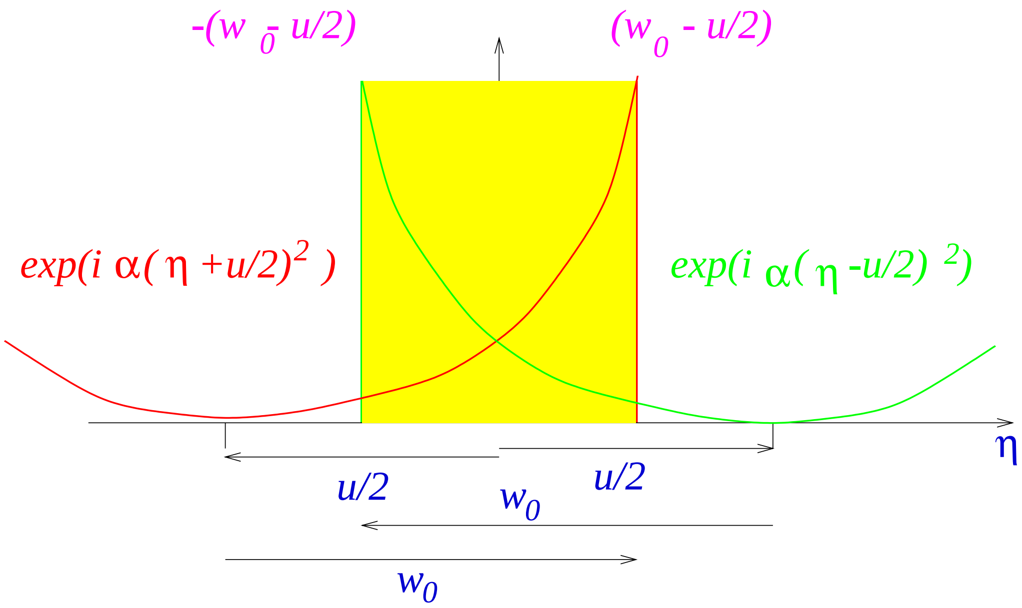
$$\alpha = \frac{\kappa\Delta W}{w_0^2} = \frac{2\pi\lambda z_1^2}{a^2}\Delta W$$

so that the CTF becomes.

$$\begin{aligned} U(u,v) &= \exp\left(\imath\alpha(u^2+v^2)\right) \quad |u|\ \&\ |v| \leq w_0 \\ &= 0 \quad \text{else} \end{aligned}$$

Now calculate the OTF in one dimension by setting $v = 0$.

Consider a shift of both CTF by $u/2$



The limits of integration are now $\pm(w_0 - \frac{u}{2})$, which is the region of overlap.

$$H(u,0) = \int_{-w_0-\frac{u}{2}}^{w_0-\frac{u}{2}} \exp\left(\imath\alpha(\eta + u/2)^2\right) \exp\left(-\imath\alpha(\eta - u/2)^2\right) d\eta$$

Expanding and canceling terms, we get

$$H(u,0) = \int_{-b}^{b} \exp(\imath\alpha 2u\eta) d\eta \quad , \quad b = w_0 - \frac{u}{2}$$

which is easily integrated to give

$$\frac{\sin(2\alpha ub)}{\alpha u} = 2b\,\mathrm{sinc}(2\alpha ub)$$

If we then normalise so that $H(0,0) = 1$, we get that

$$H(u,0) = \frac{b}{w_0}\mathrm{sinc}(2\alpha ub)$$

Cont:
Note that

$$\frac{b}{w_0} = \left(1 - \frac{u}{2w_0}\right) = H_0(u,0)$$

which is the OTF without defocus. We get that

$$H(u,0) = H_0(u,0)\,\mathrm{sinc}(2\alpha bu)$$

Noting that $\mathrm{sinc}() \leq 1$ so that

$$H(u,0) \leq H_0(u,0)$$

which says that the OTF with defocus is ALWAYS worse (lower) than the OTF at focus.

Two Dimensional Expression

The full two dimensional is just the product of the one dimensional case, that being

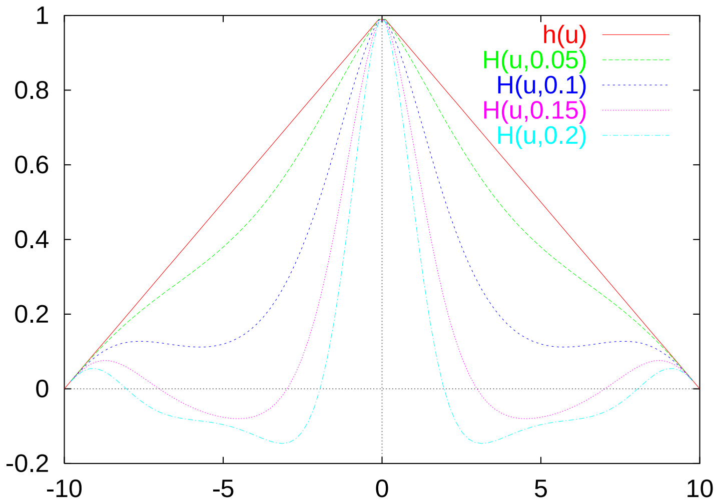
$$H(u,v) = H_0(u,v)\,\mathrm{sinc}(2\alpha bu)\,\mathrm{sinc}(2\alpha cv)$$

where

$$b = w_0 - \frac{u}{2} \quad \& \quad c = w_0 - \frac{v}{2}$$

Shape of OTF

For small defocus, OTF is reduced at high spatial frequencies, but problem for large defocus that OTF can go NEGATIVE



Graph for $\alpha = 0, 0.05, 0.1, 0.15, 0.2$.

So Zeros will occur if

$$2\alpha u b > \pi \quad \text{for } 0 \leq u \leq 2w_0$$

Noting that

$$\alpha = \frac{\kappa \Delta W}{w_0^2}$$

we get that

$$2\alpha u b = 4\kappa\Delta W \left(\frac{u}{v_0}\right)\left(1 - \frac{u}{v_0}\right)$$

where $v_0 = 2w_0$. So zeros will occur if

$$4\kappa\Delta W \left(\frac{u}{v_0}\right)\left(1 - \frac{u}{v_0}\right) > \pi \quad \text{for } 0 \leq u \leq v_0$$

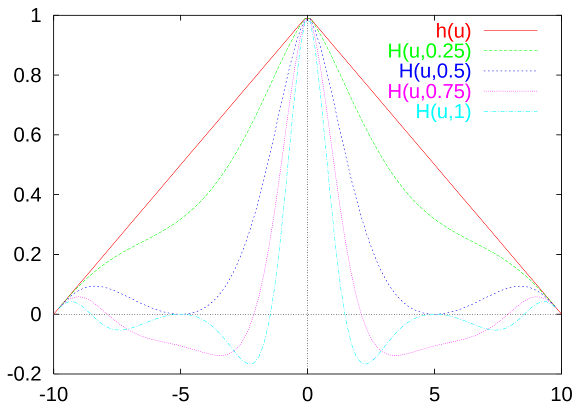
Cont:
The maximum occurs at $u = \frac{1}{2}v_0$, so we get zeros iff

$$\kappa \Delta W \geq \pi$$

which will occur is the defocus term

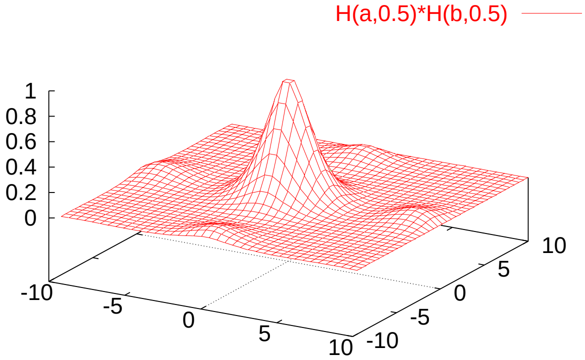
$$\Delta W \geq \lambda/2 \quad [0.63\lambda \quad \text{Round Lens}]$$

Note: The Strehl limit was $\Delta W < \lambda/4$, so zeros in the OTF start to occur at about TWICE the Strehl limit.



Plot of OTF for $\Delta W = 0, \lambda/4, \lambda/2, 3\lambda/4, \lambda$

In two dimensions we get the produce of the OTF, so giving for $\Delta W = \lambda/2$



Negative OTF

What does Negative OTF regions mean?

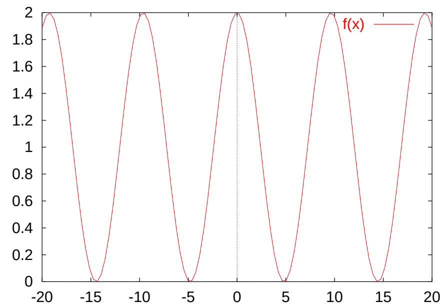
If we have that

$$f(x,y) = 1\cos(2\pi a x)$$

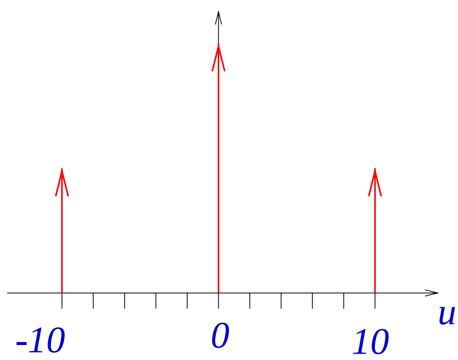
then this FT to get

$$F(u,v) = \delta(0) + \frac{1}{2}\delta(u+a) + \frac{1}{2}\delta(u-a)$$

so for $a = 10$ we get:



Input Function $f(x)$



Fourier Transform $F(u)$

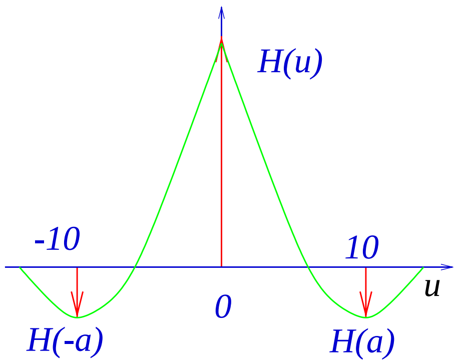
If at that frequency, $H(a) = -A$, then

$$G(u,v) = \delta(0) - A\frac{1}{2}\delta(u+a) - A\frac{1}{2}\delta(u-a)$$

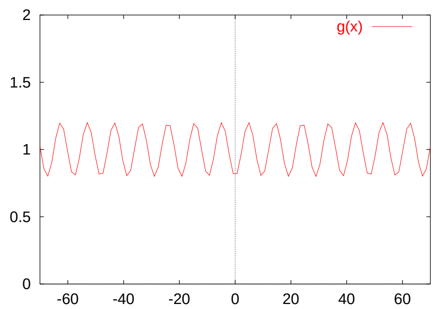
so that the output is given by

$$g(x,y) = 1 - A\cos(2\pi a x)$$

which is a grating of the same spatial frequency, but with the Contrast Reversed and reduced to A.



Fourier $F(u)H(u)$



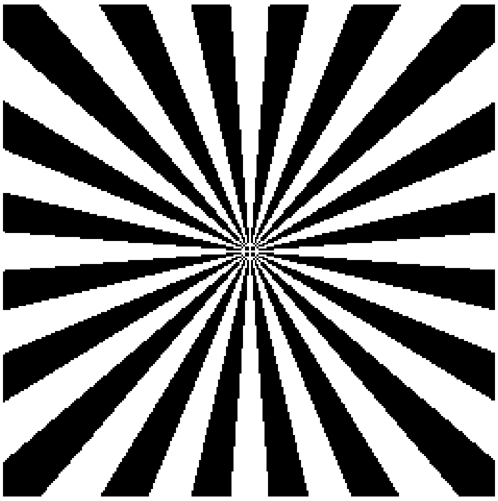
Output $g(x)$

So at large defocus we get **Contrast Reversal** at certain spatial frequencies. (See Goodman page 150, figure 6.12)

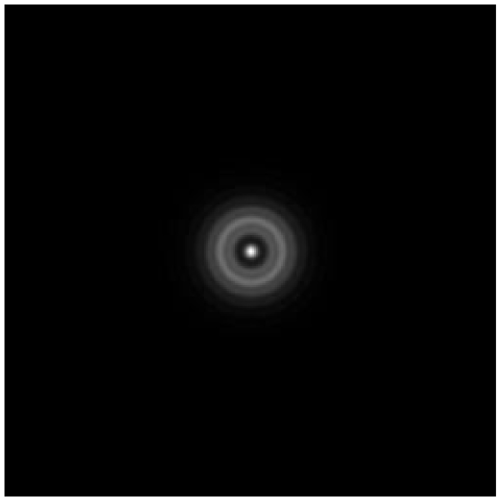
This is large defocus and results in a very poor image.

Digital Defocus Example

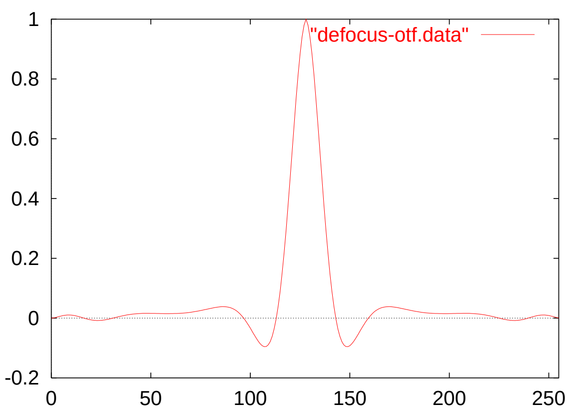
Example of defocus of $\Delta W = 3/2\lambda$, ($6\times$ Strehl limit).



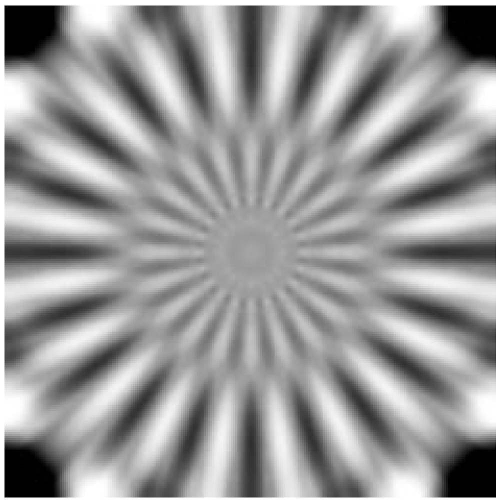
Input image $f(x,y)$



Defocused PSF



X-Section of OTF



Output Image (enhanced)