Tutorial Solutions

2 Scalar Diffraction

The first two questions explore the required assumptions to obtain Fresnel diffraction from the full scalar theory. The second question looks at the practical implications of these approximations and although rather difficult, should be worked through. The algebra in the last part of the second question is rather messy, and while the details are beyond this course the final result is very important.

The next three questions detail the three solvable cases in Fresnel diffractions. The details of these are beyond this course they are included here for completeness and you are encouraged to look at them.

2.1 Approximations

The full Rayleigh-Sommerfeld diffraction equation for propagation between two planes P_0 and P_1 separated by a distance z is given by:

$$u(x,y;z) = \frac{1}{\lambda}\int\int u_0(s,t)\left[\frac{1}{\kappa l} - \imath\right]\frac{z}{l}\frac{\exp(\imath\kappa l)}{l}\mathrm{d}s\mathrm{d}t$$

where $u_0(x,y)$ is the amplitude distribution in P_0, $u(x,y;z)$ the amplitude distribution in P_1, and $l^2 = (x-s)^2 + (y-t)^2 + z^2$.

Derive an expression for the amplitude distribution $u(x,y;z)$ assuming the Fresnel Approximation stating the approximations make. Show that this can be written as a convolution the form

$$u(x,y;z) = u_0(x,y) \odot h(x,y;z)$$

where $h(x,y;z)$ is the Fresnel free space propagation function.

This can then be written, in Fourier Space, as

$$U(u,v;z) = U_0(u,v)\,H(u,v;z).$$

Derive an expression the $H(u,v;z)$

Hint: You will need to use the result "Fourier Transform; (What you need to know)"

Solution

Firstly take the Kirchoff approximation which is valid if the planes are spearated by more than "a-few" wavelengths (see next solution for examples). Look at the term in [], which we can write at

$$\left[\frac{\lambda}{2\pi l} - \imath\right]$$

so noting that $l > z$ then if $z \gg \lambda$ then this term $\to -\imath$, giving the Kirchoff approximation of

$$u(x,y;z) = \frac{1}{\imath\lambda}\int\int u_0(s,t)\frac{z}{l}\frac{\exp(\imath\kappa l)}{l}\mathrm{d}s\mathrm{d}t$$

For the Fresnel approximation we take the paraxial approximation, so we assume that P_0 and P_1 are "small" and well separated, (see next solution for numerical examples). Consider multiplicative term in the integral, which we can write as:

$$\frac{z}{l^2}\exp(\imath\kappa l)$$

and, if we let $a = x - s$ and $b = y - t$, we can write

$$l = z\left(1 + \frac{a^2+b^2}{z^2}\right)^{1/2}$$

If the planes are small and well separated, then $a, b \ll z$, so we can take Taylor expansion of l, as follows:

Amplitude Term: Take *Zero Order* approximation, so

$$\frac{z}{l^2} \approx \frac{1}{z} = \text{Constant}$$

Phase Term: Take *First Order* approximation, so

$$\kappa l \approx \kappa z + \frac{\kappa}{2z}(a^2+b^2)$$

Note: you need to take the *First Order* approximation for the phase term since it is multiplied by κ which is typically a *large* number.

Taking these two approximation, we get the Fresnel Approximation of

$$u(x,y;z) = \frac{\exp(\imath\kappa z)}{\imath\lambda z}\iint u_0(s,t)\exp\left(\imath\frac{\kappa}{2z}\left((x-s)^2+(y-t)^2\right)\right)\mathrm{d}s\mathrm{d}t$$

If we note that the convolution integral is given by

$$f(x,y)\odot g(x,y) = \iint f(s,t)\, g((x-s),(y-t))\,\mathrm{d}s\mathrm{d}t$$

then the above expression is obviously a convolution with

$$u(x,y;z) = u_0(x,y)\odot h(x,y;z)$$

where we have that

$$h(x,y;z) = \frac{\exp(\imath\kappa z)}{\imath\lambda z}\exp\left(\imath\frac{\kappa}{2z}(x^2+y-^2)\right)$$

which is the Fresnel Free Space Progagation function stated in Lectures.

Since this in a convolution in real space, then in Fourier space we have that

$$U(u,v;z) = U_0(u,v)\, H(u,v;z).$$

where $H(u,v;z) = \mathcal{F}\{h(x,y;z)\}$, which noting that thr Fourier Transform in linear, we can write as

$$H(u,v;z) = \frac{\exp(\imath\kappa z)}{\imath\lambda z}\int\exp\left(\imath\frac{\kappa}{2z}x^2\right)\exp(-\imath 2\pi ux)\mathrm{d}x\int\exp\left(\imath\frac{\kappa}{2z}y^2\right)\mathrm{d}x\exp(-\imath 2\pi vy)\mathrm{d}y$$

so we need only evaluate one of the integrals.
Noting the result given that

$$\int_{-\infty}^{\infty} \exp(-bx^2)\exp(iax)\mathrm{d}x = \sqrt{\frac{\pi}{b}}\exp\left(-\frac{a^2}{4b}\right)$$

(See "Mathematical Handbook", MR Spiegel, McGraw-Hill, Page 98, Definite Integral 15.73. The given identy is actually,

$$\int_{0}^{\infty} \exp(-bx^2)\cos(ax)\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{b}}\exp\left(-\frac{a^2}{4b}\right)$$

but this can be extended to the $\infty \to \infty$ exp() integral required by noting that the cos() is symmetric so $-\infty \to \infty$ integral is double the $0 \to \infty$ integral and that sin() is anti-symmetric to the $\infty \to \infty$ integral is zero.)
If we let $b = \frac{-\imath\kappa}{2z}$ and $a = 2\pi u$ then

$$\int \exp\left(\imath\frac{\kappa}{2z}x^2\right)\exp(-\imath 2\pi u x)\mathrm{d}x = \sqrt{\imath\frac{2z\pi}{\kappa}}\exp\left(\imath\frac{2z\pi^2}{\kappa}u^2\right) = \sqrt{\imath\lambda z}\exp\left(\imath\pi z\lambda u^2\right)$$

We get the same expression for the y integral, so multiplying these together we get,

$$H(u,v;z) = \exp(\imath\kappa z)\exp\left(\imath\pi z\lambda(u^2+v^2)\right)$$

which is also a quadratic phase term.
Note u, v are Spatial Frequency, so have dimensions of m^{-1} Both exp() terms thus have dimensionless operands (as would be expected!).

2.2 Intensity Variations

For a point source of amplitude A located at $(0,0)$ in plane P_0, calculate the expression for the the intensity in plane P_1 which is parallel to P_0 and separated by a distance z using.

1. the Rayleigh-Sommerfeld relation,
2. the Kirchoff approximation,
3. the Fresnel approximation.

Show that the *maximum* intensity difference between Rayleigh-Sommerfeld and Kirchoff expressions occurs on-axis and calculate the distance between the planes when the two results differ by i) 1%, ii) 2% and iii) 5%.

Using MAPLE or GNUPLOT plot the intensity pattern when the planes are separated by *one* wavelength for the Rayleigh-Sommerfeld *and* the Kirchoff approximation. Is this plot consistent with your answer above?

Show that the fractional difference between intensity predicted by the Fresnel and Kirchoff approximations is zero on-axis,and and is given by

$$\varepsilon \approx 2\theta^2$$

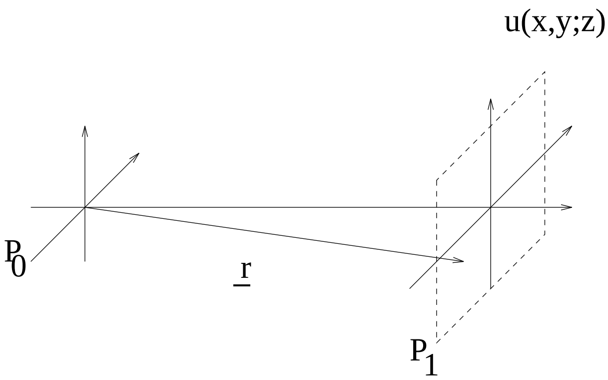
there θ is the off-axis angle. Calculate the maximum angle for an error of i) 1%, ii) 2% and iii) 5%.

Solution

For a point at position $(0,0)$ in plane P_0 then in plane we have

$$u_0(x,y) = A\delta(x,y)$$

so that P_1 we get,



$$u(x,y;z) = u_0(x,y) \odot h(x,y;z) = A\,h(x,y;z)$$

where $h(x,y;z)$ is the *Free Space Propagation Function*, so the intensity in P_1 is

$$g(x,y;z) = |A\,h(x,y;z)|^2$$

Rayleigh-Sommerfeld: the full solution is given by

$$h_R(x,y;z) = \frac{1}{\lambda}\left[\frac{1}{\kappa r} - \imath\right]\frac{z}{r}\frac{\exp(\imath\kappa r)}{r}$$

so the intensity

$$g_R(x,y;z) = A^2\frac{z^2}{\lambda^2 r^4}\left[1 + \frac{\lambda^2}{4\pi^2 r^2}\right]$$

where $r^2 = x^2 + y^2$.

Kirchoff Approximation: we have that

$$h_K(x,y;z) = \frac{1}{\imath\lambda}\frac{z}{r}\frac{\exp(\imath\kappa r)}{r}$$

so the intensity is:

$$g_K(x,y;z) = A^2\frac{z^2}{\lambda^2 r^4}$$

Fresnel Approximation: we have that

$$h(x,y;z) = \frac{\exp(\imath\kappa z)}{\imath\lambda z}\exp\left(\imath\frac{\kappa}{2z}(x^2+y^2)\right)$$

so that th intensity

$$g(x,y;z) = A^2\frac{1}{\lambda^2 z^2} = \text{Constant}$$

Part b: The difference between the Rayleigh-Sommerfeld and Kirchoff intensity is given by

$$\Delta g(x,y;z) = g_R(x,y;z) - g_K(x,y;z) = A^2 \frac{z^2}{4\pi^2 r^6}$$

For a given plane separation z, since $r^2 = x^2 + y^2 + z^2$, then the r^6 is a **minimum** when $x = y = 0$. So that Δg is a **maximum** when $x = y = 0$, so that the biggest difference occurs on the optical axis.

The fractional difference on-axis is given by

$$\frac{\Delta g(0,0;z)}{g_K(0,0;z)} = \frac{\lambda^2}{4\pi^2 z^2}$$

For a 1% error:

$$\frac{\lambda^2}{4\pi^2 z^2} = \frac{1}{100} \quad \Rightarrow \quad z = \frac{10}{2\pi}\lambda = 1.59\lambda$$

For a 2% error:

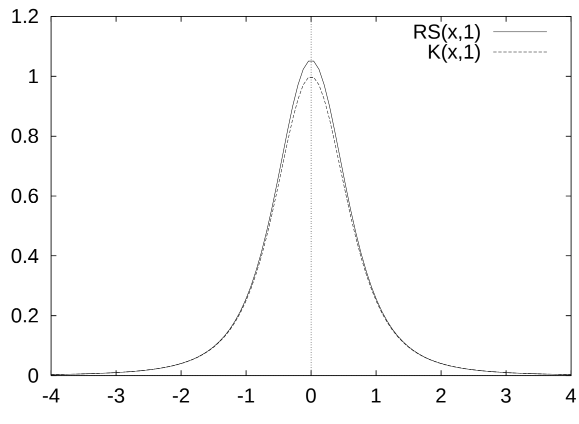
$$\frac{\lambda^2}{4\pi^2 z^2} = \frac{1}{50} \quad \Rightarrow \quad z = \frac{5}{\sqrt{2}\pi}\lambda = 1.12\lambda$$

For a 5% error:

$$\frac{\lambda^2}{4\pi^2 z^2} = \frac{1}{20} \quad \Rightarrow \quad z = \frac{\sqrt{5}}{\pi}\lambda = 0.71\lambda$$

So provided that the planes are separated by more than a "few wavelengths" the difference between the Rayleigh-Sommerfeld and the Kirchoff approximations is very small, and in almost all cases can be ignored.

Part c: With a plane separation of one wavelength, then $z = \lambda$. We then want to plot the two functions $g_R(x,0;\lambda)$ and $g_K(x,0;\lambda)$. It is also convenient to set $\lambda = 1$ then $z = 1$ and x-axis has units of wavelengths. The plot using, in this case GNUPLOT is:



where $RS(x,1)$ is the Rayleigh-Sommerfeld and $K(x,1)$ is the Kirchoff. We see from this plot that the Rayleigh-Sommerfeld is slightly "taller and thinner" than the Kirchoff with the largest difference as $x = 0$ which is the "on-axis" point. This is in exact agreement with the above analysis.

Aside: if you repeat this plot for $z > 3\lambda$ then the two plot are essentially identical. This is further confirmation that provided that the planes are separated by more than a "few" wavelengths the Kirchoff approximation is good enough in all practical situations.

Part d: the difference between the Kirchoff and Fresnel approximation is given by

$$\Delta g(x,y;z) = g_K(x,y;z) - g_F(x,y;z) = A^2 \frac{1}{\lambda^2 z^2} \left[1 - \frac{z^4}{r^4}\right]$$

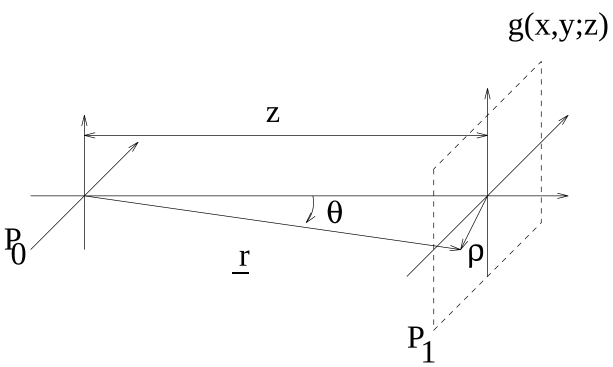
Note that on-axis, where $x = y = 0$ so $r = z$ then the both approximations give the same answer. The fractional change is thus given by

$$\frac{\Delta g(x,y;z)}{g_F(x,y;z)} = \left[1 - \frac{z^4}{r^4}\right]$$

So for a given error ε, we have that

$$\varepsilon = \left[1 - \frac{z^4}{r^4}\right] \quad \Rightarrow \quad r^2 = \frac{z^2}{\sqrt{1-\varepsilon}}$$

We have that $r^2 = x^2 + y^2$, so if we define $\rho^2 = x^2 + y^2$, we have that $r^2 = \rho^2 + z^2$,



so we get that

$$\rho^2 = \frac{z^2}{\sqrt{1-\varepsilon}} - z^2$$

which can be written as:

$$\rho^2 = z^2 \left[\frac{1}{\sqrt{1-\varepsilon}} - 1\right]$$

If we assume that $\rho \ll z$ then we have that

$$\theta \approx \frac{\rho}{z} = \sqrt{\frac{1}{\sqrt{1-\varepsilon}} - 1}$$

and additionally, if we assume that $\varepsilon \ll 1$ we can expand the inner square root to get

$$\theta \approx \sqrt{\frac{\varepsilon}{2}}$$

so that

$$\varepsilon \approx 2\theta^2$$

At 1% error: $\varepsilon = 0.01$, so that

$$\theta_{max} = 0.07 \text{ Radians} = 4^\circ$$

At 2% error: $\varepsilon = 0.02$, so that

$$\theta_{max} = 0.1 \text{ Radians} = 5.73^\circ$$

At 5% error: $\varepsilon = 0.05$, so that

$$\theta_{max} = 0.161 \text{ Radians} = 9.2^\circ$$

So the Fresnel approximation is valid on-axis and for a few degrees off-axis but the error rises with the square of the angle. Useful until about $\pm 10^\circ$ off-axis. Note this approximation is a much greater problem since there are *many* optical systems with a field of view greater than $\pm 10^\circ$.

2.3 Fresnel Diffraction from a Rectangular Aperture

Analytic solutions for Fresnel diffraction are few and far between, and even the simple cases result in fairly horrible mathematical manipulation and special functions. This is one of the few reasonable tractable cases!

Consider a rectangular aperture of size $2a \times 2b$ in plane P_0 illuminated by a collimated beam of wavelength λ. Show that the Fresnel approximation gives and *intensity* in a plane P_1 a distance z by:

$$I(x,y;z) = \frac{1}{4}\left\{|C(\alpha_2) - C(\alpha_1)|^2 + |S(\alpha_2) - S(\alpha_1)|^2\right\} \times \left\{|C(\beta_2) - C(\beta_1)|^2 + |S(\beta_2) - S(\beta_1)|^2\right\}$$

where

$$\alpha_1 = \sqrt{\frac{2}{\lambda z}}(x+a)$$
$$\alpha_2 = \sqrt{\frac{2}{\lambda z}}(x-a)$$
$$\beta_1 = \sqrt{\frac{2}{\lambda z}}(y+b)$$
$$\beta_2 = \sqrt{\frac{2}{\lambda z}}(y-b)$$

$C(p)$ and $S(p)$ are the *Fresnel Integrals* given by

$$C(p) = \int_0^p \cos\left(\frac{\pi}{2}u^2\right) \mathrm{d}u \quad \text{and} \quad S(p) = \int_0^p \sin\left(\frac{\pi}{2}u^2\right) \mathrm{d}u$$

Hint: The both Fresnel Integrals are anti-symmetric (odd), so that $C(-p) = -C(p)$ and $S(p) = -S(p)$.

Solution

Take the rectangular aperture of size $2a \times 2b$ illuminated with a collimated beam to be in plane P_0, so the amplitude $u_0(x,y)$ is given by,

$$\begin{aligned} u_0(x,y) &= 1 \quad \text{for } |x| < a \text{ and } |y| < b \\ &= 0 \quad \text{else} \end{aligned}$$

so the amplitude distribution in P_1 a distance zm from the eqation at the top of slide 18 is,

$$u(x,y,z) = \frac{\exp(\imath\kappa z)}{\imath\lambda z}\int_{-b}^{b}\int_{-a}^{a}\exp\left[\frac{\imath\kappa}{2z}\left((x-s)^2+(y-t)^2\right)\right]\,\mathrm{d}s\,\mathrm{d}t$$

This integral is separable, (due the exp() **and** the simple shape of the aperture), so can be written as:

$$u(x,y,z) = \frac{\exp(\imath\kappa z)}{\imath\lambda z}\int_{-a}^{a}\exp\left(\frac{\imath\kappa}{2z}(x-s)^2\right)\,\mathrm{d}s\int_{-b}^{b}\exp\left(\frac{\imath\kappa}{2z}(y-t)^2\right)\,\mathrm{d}t$$

Look at one of these integrals and expand the exp() in terms of cos() and sin() to get,

$$\int_{-a}^{a}\exp\left(\frac{\imath\kappa}{2z}(x-s)^2\right)\,\mathrm{d}s = \int_{-a}^{a}\cos\left(\frac{\kappa}{2z}(x-s)^2\right)\,\mathrm{d}s + \imath\int_{-a}^{a}\sin\left(\frac{\kappa}{2z}(x-s)^2\right)\,\mathrm{d}$$

Now substitues

$$p = \sqrt{\frac{2}{\lambda z}}(x-s)$$

so the limits of integration become

$$\begin{aligned}-a &\rightarrow \alpha_1 = \sqrt{\frac{2}{\lambda z}}(x+a)\\ a &\rightarrow \alpha_2 = \sqrt{\frac{2}{\lambda z}}(x-a)\end{aligned}$$

so giving the above integral as

$$\sqrt{\frac{\lambda z}{2}}\int_{\alpha_1}^{\alpha_2}\cos\left(\frac{\pi}{2}p^2\right)\,\mathrm{d}p + \imath\sqrt{\frac{\lambda z}{2}}\int_{\alpha_1}^{\alpha_2}\sin\left(\frac{\pi}{2}p^2\right)\,\mathrm{d}p$$

using the fact that the $\cos(u^2)$ and $\sin(u^2)$ integrals are odd, finially, get that

$$A\left[C(\alpha_2)-C(\alpha_1)\right] + \imath A\left[S(\alpha_2)-S(\alpha_1)\right]$$

where $C()$ and $S()$ are the Fresnel integrals and $A = \sqrt{\lambda z/2}$.
The integral is the t direction is identical, so with

$$\begin{aligned}\beta_1 &= \sqrt{\frac{2}{\lambda z}}(y+b)\\ \beta_2 &= \sqrt{\frac{2}{\lambda z}}(y-b)\end{aligned}$$

we simply get it to be:

$$A\left[C(\beta_2)-C(\beta_1)\right] + \imath A\left[S(\beta_2)-S(\beta_1)\right]$$

so finally we can substitute these two solutions to get

$$\begin{aligned}u(x,y;z) = B\left\{\left[C(\alpha_2)-C(\alpha_1)\right] + \imath\left[S(\alpha_2)-S(\alpha_1)\right]\right\}\times\\ \left\{\left[C(\beta_2)-C(\beta_1)\right] + \imath\left[S(\beta_2)-S(\beta_1)\right]\right\}\end{aligned}$$

where

$$B = A^2\frac{\exp(\imath\kappa z)}{\imath\lambda z} = \frac{\imath}{4}\exp(\imath\kappa z)$$

The intensity is now just the square modulus of $u(x,y;z)$ which is

$$I(x,y;z) = \frac{1}{4}\left\{|C(\alpha_2)-C(\alpha_1)|^2 + |S(\alpha_2)-S(\alpha_1)|^2\right\}\times\left\{|C(\beta_2)-C(\beta_1)|^2 + |S(\beta_2)-S(\beta_1)|^2\right\}$$

as required.

2.4 Fresnel Diffraction from One dimensional Objects

Using the above expression for the Fresnel approximation (whether you can prove it or not!) for the intensity diffracted from a rectangle in plane P_0 derive expressions for the intensity diffracted into plane P_1 when the object in plane P_0 is:

1. A long thin slit of width $2a$.
2. A "knife-edge" at $y = 0$.

where the separation between the planes is z.

Hint: You will have to "look-up-in-a-book" what happens to $C(p)$ and $S(p)$ as $p \to \pm\infty$!

The two Fresnel integrals have no analytical solutions but can be numerically calculated relatively simply and efficiently depending on the size of $|p|$ by either series summation or continuing fractions. There are two demonstration programs located at:

```
~wjh/mo/examples/edge
```

and

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~wjh/mo/examples/slit
```

that solve these numerically. Use these to explore the effects of Fresnel diffraction for slits and edges. (There is no programming, just run the programs and respond to the prompts.)

Both these programs assume HeNe wavelength (633 nm) and produce graphs with 1024 points which can be optionally saved as encapsulated Postscript.

Aside: You *can* do this calculation "manually" by using the Cornu Spiral, which is a plot of $S(p)$ against $C(p)$, and a ruler, but you should be using late 20th century technology and methods rather than tables and graphs from 18th century Frenchmen! See Guenther, Appendix 11-B, page 468 if you must!

Solution

1) With a slit of width $2a$ the amplitude distribution in P_0 is

$$\begin{aligned} u_0(x,y) &= 1 \quad \text{for } |x| < a \\ &= 0 \quad \textit{else} \end{aligned}$$

or alternatively it is a rectangle with the x dimension of $2a$ and the y dimension being $b = \infty$. Putting this in the above relation for the intensity diffracted from a rectangular aperture we have

$$\beta_1 = \infty \quad \text{and} \beta_2 = -\infty$$

so noting, from standard maths books, we have that

$$C(\infty) = S(\infty) = 0.5 \quad \text{and} \quad C(-\infty) = S(-\infty) = -0.5$$

then the intensity in plane P_0 is then

$$I(x;z) = \frac{1}{2}\left\{ |C(\alpha_2) - C(\alpha_1)|^2 + |S(\alpha_2) - S(\alpha_1)|^2 \right\}$$

where, from above

$$\alpha_1 = \sqrt{\frac{2}{\lambda z}}(x+a)$$
$$\alpha_2 = \sqrt{\frac{2}{\lambda z}}(x-a)$$

2) With a "knife-edge" at $x = 0$ the amplitude distribution in P_0 is

$$\begin{aligned} u_0(x,y) &= 1 \quad \text{for } x > 0 \\ &= 0 \quad else \end{aligned}$$

but this can be also be considered as a slit with one side at $x = 0$ and the other side at ∞. Now noting that the two α(s) above have terms of the form $(x \pm a)$ they are recognised as being associated with the location of the two edges of the slit. In particular, by looking at the limits of integration in the previous solution, we had an integration form $-a \to a$ (limits of the rectangle) that was transformed (by substitution) to an integral with limits $\alpha_1 \to \alpha_2$. This allows us to recognise that

$$\begin{aligned} \alpha_1 &\Rightarrow \text{Due to edge at } x = -a \\ \alpha_2 &\Rightarrow \text{Due to edge at } x = a \end{aligned}$$

So to form the "knife-edge" we move the slit edge from $a \to \infty$, and from $-a \to 0$ we have that

$$\begin{aligned} \alpha_1 &= \sqrt{\frac{2}{\lambda z}}x \\ \alpha_2 &= -\infty \end{aligned}$$

so by simply substituting into the above intensity relation for a slit we get the intensity diffracted from a "knife-edge" to be

$$I(x;z) = \frac{1}{2}\left\{|C(\alpha_1)+0.5|^2 + |S(\alpha_1)+0.5|^2\right\}$$

Numerical Simulations:

The Fresnel approximation is valid for,

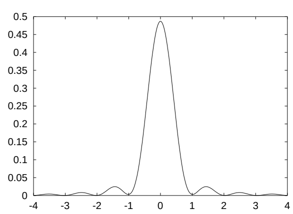
1. Feature size of object $\gg \lambda$, for for HeNe light slit should be wider than about $30\,\mu$m which is about $50\times\ \lambda$. There is no restriction for the knife edge.

2. Plane separation must be $\gg \lambda$. (Not a significant restriction).

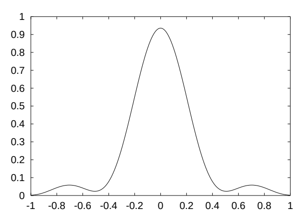
3. Dimensions in plane P_0 and $P_1 \ll z$, so minimum plane separation should be about $20\times$ width of the slit.

Typical results:

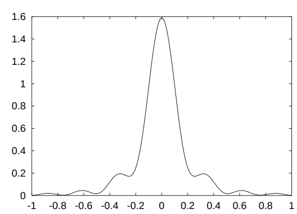
Slit: of width 0.5 mm, wavelength of 633 nm (HeNe laser) and distances as shown,



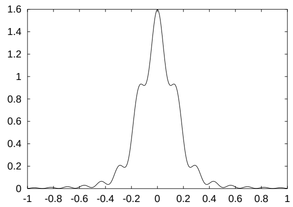
800 mm



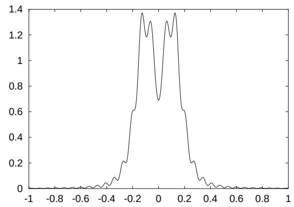
400mm



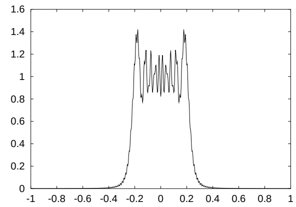
200mm



100 mm



50mm



10mm

These results show that for very large z ($> 800\,\text{mm}$) then the intensity pattern is the $\text{sinc}()^2$ expected for far field diffraction, while for small(ish) ($z = 10\,\text{mm}$) distances we get essentially the same shape as the slit but with high frequency “interference” fringes.

Note for small(ish) z the values of α_1 and α_2 become large and the $C()$ and $S()$ integrals become highly oscillators and subject to large numerical calculation errors.

Aside: Numerical calculation of any diffraction effects is littered with numerical problems and in fact calculation of Fresnel diffraction for arbitrary shaped two-dimensional apertures is still an active topic in computational physics. The calculation of Kirchoff diffraction is far worse since there are no separable solutions and direct calculation involves the summation of a large numbers of highly oscillatory term. Trying do to the calculation in Fourier space looks attractive but in fact is just as problematical!

2.5 Poisson Spot

Poisson’s Spot is one of the great results of classical physics. The aim is to use Fresnel diffraction to calculate the intensity pattern “on-axis” behind an opaque disc of radius a when it is illuminated by a collimated beam.

To perform this you must:

1. Calculate the amplitude “on-axis” in plane P_1 ie, $u(0,0;z)$ when the object in plane P_0 is a transparent disc of radius a.

2. Use Babinet’s principle which states that the and amplitude diffracted from a opaque object is the amplitude if no object is present $-$ the amplitude diffracted from the equivalent transparent object. *Note: this is a direct consequence of the Helmholtz equation, from what Fresnel diffraction was derived, being linear!*

You will get a surprising result!

Solution

Start with a circular object in plane P_0 of radius a, so the amplitude in this plane is just,

$$\begin{aligned} u_0(x,y) &= 1 \quad \text{for } x^2+y^2 \le a^2 \\ &= 0 \quad \text{else} \end{aligned}$$

Fresnel diffraction to plane P_1 a distance z away is given by,

$$u(x,y;z) = \frac{\exp(\imath\kappa z)}{\imath\lambda z} \iint u_0(s,t) \exp\left[\frac{\imath\kappa}{2z}\left((x-s)^2+(y-t)^2\right)\right] \mathrm{d}s\,\mathrm{d}t$$

We are dealing with a cylindrical system so the area of integration is much simpler in polar coordinates of

$$s = \rho\cos\theta \quad \text{and} \quad t = \rho\sin\theta$$

which, on-axis, at $x = y = 0$, we get,

$$u(0,0;z) = \frac{\exp(\imath\kappa z)}{\imath\lambda z}\int_0^{2\pi}\int_0^a \exp\left[\frac{\imath\kappa}{2z}\rho^2\right]\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

There is no θ dependence, so the θ integral simply results in a 2π constant, giving,

$$u(0,0;z) = \frac{2\pi\exp(\imath\kappa z)}{\imath\lambda z}\int_0^a \exp\left[\frac{\imath\kappa}{2z}\rho^2\right]\rho\,\mathrm{d}\rho$$

Aside: The integral

$$\int_0^r \exp(\imath r^2)\,r\,\mathrm{d}r = \frac{1}{2}\left(1 - \imath\exp(\imath r^2)\right)$$

which you can, fairly easily, shown using integration by substitution of $t = r^2$.

Now letting

$$s = \sqrt{\frac{\kappa}{2z}}\rho$$

we get that

$$u(0,0;z) = \frac{2\pi\exp(\imath\kappa z)}{\imath\lambda z}\left(\frac{2z}{\kappa}\right)\int_0^{s_0}\exp(\imath s^2)\,s\,\mathrm{d}s$$

where $s_0 = \sqrt{\kappa/2z}a$, which, from above gives,

$$u(0,0;z) = -\imath\exp(\imath\kappa z)\left(1 - \imath\exp\left(\imath\frac{\kappa a^2}{2z}\right)\right)$$

The intensity is the square modulus of this, which gives, after a bit of tricky manipulation, that,

$$I(0,0;z) = 2\cos^2\left(\frac{\kappa a^2}{2z}\right)$$

so you set a series of bright and dark spots in the centre.

Babinet's principle states that the diffraction in plane P_1 from an opaque object in plane P_1 will be the amplitudes of there is no present − the amplitude for the transparent version of the object.

If there is no object in plane P_0 then in plane P_1 after the light have traveled a distance z, the amplitude will be

$$v(x,y;z) = \exp(\imath\kappa z)$$

so behind the disc, on-axis, the amplitude distribution will be

$$\exp(\imath\kappa z) + \imath\exp(\imath\kappa z)\left(1 - \imath\exp\left(\imath\frac{\kappa a^2}{2z}\right)\right)$$

The intensity is the square modulus of this, which as above, after some rather tricky manipulation, gives the result that:

$$I(0,0;z) = 2\cos^2\left(\frac{\kappa a^2}{2z}\right)$$

so again a series of “bright spots” on-axis, which is a little “odd”!

Historical background: The result was first derived by Poisson in an attempt to discredit the wave theory of light but forward by Fresnel. Poisson was convinced by the Newtonian particle theory of light. The argument was that the bright spots predicted behind a opaque circular disc were “contrary of common sense”. Initially, in a crude experiment, Poisson did not find these spots. However a later more careful experiment showed then. This discovery was a verification of Fresnel theory which is one of the cornerstones of classical physics. It is direct experimental proof of the wave theory of light and could not be explained by any “particle” based theory. This type of results is not particularly surprising is view of Maxwell equations, but remember that Fresnel and Poisson pre-date Maxwell by about 50 years.