Topic 2: Scalar Diffraction Theory

Aim

These two letures develop Scalar Diffraction Theory. Scalar wave theory is initially used to derive the general Rayleigh-Sommerfeld diffraction relation, which is then approximated to the Kirchhoff and finally the Fresnel diffraction approximations. These relations will then be applied to optical systems in the subsequent lectures.

References

- Goodman, Chapter 3 & **Chapter 4**
  (Chapter 3 this gives and alternative derivation that is **NOT** examinable).
- Guenther, Appendix 9-A & 9-B.
- Born & Wolf, Chapter 8.1 to 8.4. (Very tough explanation)

2 Scalar Diffraction

The first two questions explore the required assumptions to obtain Fresnel diffraction from the full scalar theory. The second question looks at the practical implications of these approximations and although rather difficult, should be worked through. The algebra in the last part of the second question is rather messy, and while the details are beyond this course the final result is very important.

The next three questions detail the three solvable cases in Fresnel diffractions. The details of these are beyond this course they are included here for completeness and you are encouraged to look at them.

2.1 Approximations

The full Rayleigh-Sommerfeld diffraction equation for propagation between two planes P_0 and P_1 separated by a distance z is given by:

$$u(x,y;z) = \frac{1}{\lambda}\int\!\!\int u_0(s,t)\left[\frac{1}{\kappa l} - \imath\right]\frac{z}{l}\frac{\exp(\imath\kappa l)}{l}\mathrm{d}s\mathrm{d}t$$

where $u_0(x,y)$ is the amplitude distribution in P_0, $u(x,y;z)$ the amplitude distribution in P_1, and $l^2 = (x-s)^2 + (y-t)^2 + z^2$.

Derive an expression for the amplitude distribution $u(x,y;z)$ assuming the Fresnel Approximation stating the approximations make. Show that this can be written as a convolution the form

$$u(x,y;z) = u_0(x,y) \odot h(x,y;z)$$

where $h(x,y;z)$ is the Fresnel free space propagation function.

This can then be written, in Fourier Space, as

$$U(u,v;z) = U_0(u,v)\, H(u,v;z).$$

Derive an expression the $H(u,v;z)$

Hint: You will need to use the result "Fourier Transform; (What you need to know)"

2.2 Intensity Variations

For a point source of amplitude A located at $(0,0)$ in plane P_0, calculate the expression for the the intensity in plane P_1 which is parallel to P_0 and separated by a distance z using.

1. the Rayleigh-Sommerfeld relation,
2. the Kirchoff approximation,
3. the Fresnel approximation.

Show that the *maximum* intensity difference between Rayleigh-Sommerfeld and Kirchoff expressions occurs on-axis and calculate the distance between the planes when the two results differ by i) 1%, ii) 2% and iii) 5%.

Using MAPLE or GNUPLOT plot the intensity pattern when the planes are separated by *one* wavelength for the Rayleigh-Sommerfeld *and* the Kirchoff approximation. Is this plot consistent with your answer above?

Show that the fractional difference between intensity predicted by the Fresnel and Kirchoff approximations is zero on-axis,and and is given by

$$\varepsilon \approx 2\theta^2$$

there θ is the off-axis angle. Calculate the maximum angle for an error of i) 1%, ii) 2% and iii) 5%.

2.3 Fresnel Diffraction from a Rectangular Aperture

Analytic solutions for Fresnel diffraction are few and far between, and even the simple cases result in fairly horrible mathematical manipulation and special functions. This is one of the few reasonable tractable cases!

Consider a rectangular aperture of size $2a \times 2b$ in plane P_0 illuminated by a collimated beam of wavelength λ. Show that the Fresnel approximation gives and *intensity* in a plane P_1 a distance z by:

$$I(x,y;z) = \frac{1}{4}\left\{|C(\alpha_2) - C(\alpha_1)|^2 + |S(\alpha_2) - S(\alpha_1)|^2\right\} \times \left\{|C(\beta_2) - C(\beta_1)|^2 + |S(\beta_2) - S(\beta_1)|^2\right\}$$

where

$$\alpha_1 = \sqrt{\frac{2}{\lambda z}}(x+a)$$
$$\alpha_2 = \sqrt{\frac{2}{\lambda z}}(x-a)$$
$$\beta_1 = \sqrt{\frac{2}{\lambda z}}(y+b)$$
$$\beta_2 = \sqrt{\frac{2}{\lambda z}}(y-b)$$

$C(p)$ and $S(p)$ are the *Fresnel Integrals* given by

$$C(p) = \int_0^p \cos\left(\frac{\pi}{2}u^2\right)\,\mathrm{d}u \quad \text{and} \quad S(p) = \int_0^p \sin\left(\frac{\pi}{2}u^2\right)\,\mathrm{d}u$$

Hint: The both Fresnel Integrals are anti-symmetric (odd), so that $C(-p) = -C(p)$ and $S(p) = -S(p)$.

2.4 Fresnel Diffraction from One dimensional Objects

Using the above expression for the Fresnel approximation (whether you can prove it or not!) for the intensity diffracted from a rectangle in plane P_0 derive expressions for the intensity diffracted into plane P_1 when the object in plane P_0 is:

1. A long thin slit of width $2a$.
2. A "knife-edge" at $y = 0$.

where the separation between the planes is z.

Hint: You will have to "look-up-in-a-book" what happens to $C(p)$ and $S(p)$ as $p \to \pm\infty$!

The two Fresnel integrals have no analytical solutions but can be numerically calculated relatively simply and efficiently depending on the size of $|p|$ by either series summation or continuing fractions. There are two demonstration programs located at:

```
~wjh/mo/examples/edge
```

and

```
~wjh/mo/examples/slit
```

that solve these numerically. Use these to explore the effects of Fresnel diffraction for slits and edges. (There is no programming, just run the programs and respond to the prompts.)

Both these programs assume HeNe wavelength (633 nm) and produce graphs with 1024 points which can be optionally saved as encapsulated Postscript.

Aside: You *can* do this calculation "manually" by using the Cornu Spiral, which is a plot of $S(p)$ against $C(p)$, and a ruler, but you should be using late 20th century technology and methods rather than tables and graphs from 18th century Frenchmen! See Guenther, Appendix 11-B, page 468 if you must!

2.5 Poisson Spot

Poisson's Spot is one of the great results of classical physics. The aim is to use Fresnel diffraction to calculate the intensity pattern "on-axis" behind an opaque disc of radius a when it is illuminated by a collimated beam.

To perform this you must:

1. Calculate the amplitude "on-axis" in plane P_1 ie, $u(0,0;z)$ when the object in plane P_0 is a transparent disc of radius a.

2. Use Babinet's principle which states that the and amplitude diffracted from a opaque object is the amplitude if no object is present − the amplitude diffracted from the equivalent transparent object. *Note: this is a direct consequence of the Helmholtz equation, from what Fresnel diffraction was derived, being linear!*

You will get a surprising result!